

Pluralities as nothing over and above*

Draft

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Consider some people: Melesha, Nadia, and Dylan. What is the relation between them—taken together—and the individual people they comprise? For example:

1. *Is Melesha one of them whenever they exist?*

In other words, is it necessarily the case that if they exist, Melesha is one of them?

2. *Must Dylan exist whenever they exist?*

In other words, is it necessarily the case that if they exist, Dylan exists?

Or consider a natural converse.

3. *Do they exist whenever each of Melesha, Nadia, and Dylan exists?*

In other words, is it necessarily the case that if Melesha, Nadia, and Dylan each exists, they do?

Standard treatments of the modal logic of plurals have typically addressed the first two kinds of question.¹ The third, however, is vitally important: positions in the philosophy of mathematics, philosophical logic, and metaphysics depend on its answer. The goal of this paper is to articulate a general conception of pluralities which answers these and many other related questions. It is based on the following simple claim, which I call the *fundamental conception* of pluralities.²

Some things are nothing over and above the individual things they comprise

Here's the plan. In section 1, I make the fundamental conception precise. I show how it can be captured both model-theoretically and axiomatically. In section 2, I use this to shed light on debates in the philosophy of mathematics, philosophical logic, and metaphysics. Section 3 is a technical appendix.

*Acknowledgements...

¹See, for example, Uzquiano [2011] and Rumfitt [2005]. Williamson [2013] is an exception. See section 2 for discussion.

²This conception is arguably implicit in a lot of work on plurals—e.g. Boolos [1971], especially in his discussion of the cheerios—but is made explicit in Roberts [2016], Roberts [2019], and Linnebo [2016].

1 Nothing over and above

The fundamental conception says that some things are nothing over and above the individual things they comprise. In other words, it says that there is *no difference* between some things taken together and those very same things taken individually. For some, this may seem like a mysterious statement, perhaps even meaningless; for others, like a truism, trivial and inferentially inert. I will show that neither reaction is correct: the fundamental conception is both tractable and has important consequences for a number of debates in philosophy. This section will be taken up with making the conception formally tractable and the next with looking at its consequences.

First, some preliminaries. I will restrict my attention to claims in the language of plural modal logic. That language, \mathcal{L} , can be obtained from an ordinary first-order language, \mathcal{L}_0 , by adding a stock of plural variables xx, yy, zz, \dots , a relation \prec intended to express the relation that holds between an object and some things when it is one of them—so, “ $x \prec xx$ ” is well-formed and read “it _{x} is one of them _{xx} ”—and a modal operator \diamond .³ For simplicity, I will also allow the identity relation to be flanked by plural variables—so, “ $xx = yy$ ” is well-formed.⁴ My goal is thus to make the fundamental conception formally tractable in so far as it concerns claims in languages like \mathcal{L} . For all I will say, it could have consequences beyond those claims in those languages.

Let $K = \langle W, D_1, D_2, I \rangle$ be an arbitrary Kripke model where D_1 is a domain function assigning a set of ‘objects’ to each world, D_2 a domain function assigning a set of ‘pluralities’ to each world, and I an interpretation function which interprets the non-logical vocabulary of \mathcal{L}_0 together with \prec . Since it has no accessibility relation, K will validate the modal logic S5. Given a plurality X , I will denote the set of things ‘among’ X at w : $I_w(X)$ (that is, $I_w(X) = \{x : w \vDash x \prec X\}$).⁵ As it stands, no assumptions have been made about the behaviour of pluralities. There may be none—all $D_2(w)$ could be empty—or there could be many more pluralities in $D_2(w)$ than there are subsets of $\bigcup_{w \in W} D_1(w)$ and every one of them could be empty. In particular, K may not conform to the fundamental conception.

1.1 A model-theoretic characterisation

I’ll start with a model-theoretic characterisation of the fundamental conception. I’ll first highlight four simple constraints that I argue should hold whenever a Kripke model conforms to the fundamental conception. I will argue, in other words, that the constraints are *sound* for the fundamental conception. I will then argue that they are *complete*: that any Kripke model satisfying the constraints conforms to the conception. The models conforming to the fundamental conception can then be characterised as precisely those models satisfying the four constraints.

³See Linnebo [2017] for more details.

⁴Nothing hangs on this, however, and I will show in section 2 how identity between plural variables can be explicitly defined and thus eliminated.

⁵Since Kripke models are classical, they do not allow for indeterminacy. Although this simplifies matters considerably, it is a significant omission. It will turn out that the fundamental conception implies the plural comprehension schema (see below) in such models. But some have argued that indeterminacy in quantifiers and predicates can undermine the plural comprehension schema (see, for example, [Yablo, 2006, p.151-152] and [Linnebo, 2016, p.670-673]). These arguments don’t appear to undermine the other components of the fundamental conception that I will identify, however, and so a modified form of the conception could survive those arguments with a suitably restricted comprehension principle.

The first two constraints are perhaps the most interesting and concern the behaviour of pluralities across worlds.

Recall our people—Melesha, Nadia, and Dylan—and consider the first two questions I raised earlier, now in light of the fundamental conception. Is Melesha is one of them whenever they exist? And, does Dylan exist whenever they do? If they are nothing over and above the individual things they comprise and they comprise precisely Melesha, Nadia, and Dylan, then *nothing more* could be needed for them to comprise Melesha, Nadia, and Dylan and for Melesha, Nadia, and Dylan to exist than that they exist. There is no metaphysical gap between them taken together and the individuals Melesha, Nadia, Dylan. So, Melesha must be one of them whenever they exist and Dylan must exist whenever they exist.

The first constraint I want to highlight generalises this insight. It says that the individual things a plurality comprises will continue to comprise it and exist whenever it exists: where the plurality goes, so too do the things that comprise it.⁶ Formally:⁷

Downward Dependence

If $X \in D_2(w')$, then $I_w(X) \subseteq D_1(w') \cap I_{w'}(X)$

Now consider the third question in light of the fundamental conception. Do they exist whenever each of Melesha, Nadia, and Dylan exist? If they are nothing over and above the individual things they comprise and they comprise precisely Melesha, Nadia, and Dylan, then *nothing more* could be needed for them to exist than that each of Melesha, Nadia, and Dylan exist. There is no metaphysical gap between the individuals Melesha, Nadia, Dylan and them taken together. So, they must exist whenever each of Melesha, Nadia, and Dylan exist.

The second constraint I want to highlight generalises this insight. It says that a plurality exists whenever the things that comprise it exist: where the things that comprise a plurality go, so too does the plurality. Formally:⁸

Upward Dependence

If $\bigcup_{w \in W} I_w(X) \subseteq D_1(w')$, then $X \in D_2(w')$

⁶To aid readability, I will frequently use the singular “plurality”. Nothing I say hangs on this and can always be reformulated in terms of genuinely plural locutions.

⁷We could, if we like, break this constraint into two further constraints corresponding to each of the two initial questions. The first would say that a plurality comprises the same things whenever it exists,

Downward Dependence₁

If $X \in D_2(w')$, then $I_w(X) \subseteq I_{w'}(X)$

and the second would say that a plurality cannot exist without the things it comprises,

Downward Dependence₂

If $X \in D_2(w')$, then $I_w(X) \subseteq D_1(w')$

⁸Given Downward Dependence, Upward Dependence comes to the simpler claim that if $X \in D_2(w)$ and $I_w(X) \subseteq D_1(w')$, then $X \in D_2(w')$. This is because, given Downward Dependence, when $X \in D_2(w)$, the things X comprises at w are the things it comprises in any world: $I_w(X) = \bigcup_{w' \in W} I_{w'}(X)$. I’ve chosen the slightly more complicated formulation in the main text because it doesn’t rely on Downward Dependence in this way.

The last two constraints I want to highlight are somewhat standard and concern the behaviour of pluralities within worlds. The first is an extensionality principle: it says that pluralities comprising the same things are identical. Clearly, if these things and those things are nothing over and above the individual things they comprise and these things comprise the same things as those, then these *are* those. Formally:

Extensionality

If $X, Y \in D_2(w)$ and $I_w(X) = I_w(Y)$, then $X = Y$.

The final constraint is a comprehension principle: it says that given a set of objects at a world, there is a plurality comprising precisely those objects. Clearly, if some things are nothing over and above the individual things they comprise and each individual thing in \mathcal{X} exists, then *nothing more* could be needed for there to exist some things comprising the \mathcal{X} s. There is no metaphysical gap between the individual things in \mathcal{X} and them taken together. Formally:⁹

Comprehension

If $\mathcal{X} \subseteq D_1(w)$, then there is $X \in D_2(w)$ such that $\mathcal{X} = I_w(X)$

The four constraints are thus sound for the fundamental conception: any model that conforms to the conception must satisfy them. I will now argue that they are complete. But before I do that, I need a distinction.

Although the constraints tell us which things a plurality comprises when it exists—namely, by Downward Dependence, whatever it comprises in any world—they do not tell us what it comprises in worlds where it doesn't exist. Consider again our people—Melesha, Nadia, and Dylan—and take a world w in which Melesha and Nadia exist but Dylan does not. By Downward Dependence, they don't exist in w . But is Melesha still one of them in w ?

It seems to me that the fundamental conception is silent on this question. Nevertheless, there appear to be three non-ad-hoc answers. First, when a plurality fails to exist, it comprises nothing. Formally:

Nothing

If $X \notin D_2(w)$, then $I_w(X) = \emptyset$

Second, it comprises everything it would otherwise comprise. Formally:

Everything

If $X \notin D_2(w)$, then $I_w(X) = \bigcup_{w' \in W} I_{w'}(X)$

Finally, it comprises the things it would otherwise comprise which also exist. Formally:

Existence

If $X \notin D_2(w)$, then $I_w(X) = \bigcup_{w' \in W} I_{w'}(X) \cap D_1(w)$

⁹It follows from **Comprehension** that there is an empty plurality. Nothing I will say depends on this, however. If we like, we can modify **Comprehension** so that it only applies to non-empty sets and require that every plurality is non-empty. See Linnebo [2017] for discussion of this point.

In what follows, I will assume that one of these answers is correct, though I will not take a stance on which.

To show that our four constraints are complete for the fundamental conception, I will employ a “squeezing” argument.¹⁰ It rests on three claims which jointly establish that the informal notion of conforming to the fundamental conception coincides with the formal notion of satisfying the four constraints.

The first claim is that a Kripke model conforms to the fundamental conception only if it satisfies the constraints: in other words, conforming to the fundamental conception implies satisfying the constraints. This was established above. The second is that every model of a certain kind conforms to the fundamental conception: in other words, being a model of that kind implies conforming to the fundamental conception. The final claim is that every model satisfying the constraints is isomorphic to a model of that kind: in other words, satisfying the constraints implies being (isomorphic to) a model of that kind. The informal notion of conforming to the fundamental conception is thus sandwiched between two formal notions which in turn coincide. It follows that all three notions coincide. I will now argue for the last two claims.

For the second claim, consider a Kripke model $K = \langle W, D_1, I \rangle$ for a first-order language. As Boolos [1985] effectively showed, we can use K to interpret the language of plural modal logic without adding domains of pluralities. The idea is simple. We take a *plural assignment function* to be a relation R that associates with each singular variable “ x ” a unique object in $\bigcup_{w \in W} D_1(w)$ and with each plural variable “ xx ” 0 or more objects from $\bigcup_{w \in W} D_1(w)$. We then define truth at a world by adding the following clauses to the usual first-order ones.

- $w \models_R \text{“}\exists xx\varphi\text{”} \leftrightarrow \exists R' =_{\text{“}xx\text{”}} R(\forall x(R'(\text{“}xx\text{”}, x) \rightarrow x \in D_1(w)) \wedge w \models_{R'} \text{“}\varphi\text{”})$, where $R' =_{\text{“}xx\text{”}} R$ means that the plural assignment R' is just like R except perhaps in what it associates with “ xx ”.
- $w \models_R \text{“}xx = yy\text{”} \leftrightarrow \forall x(R(\text{“}xx\text{”}, x) \leftrightarrow R(\text{“}yy\text{”}, x))$

together with one of the following (depending on which of Everything, Nothing, or Existence we want to adopt),

- (E) $w \models_R \text{“}x \prec xx\text{”} \leftrightarrow R(\text{“}xx\text{”}, x)$
- (N) $w \models_R \text{“}x \prec xx\text{”} \leftrightarrow R(\text{“}xx\text{”}, x) \wedge w \models_R \text{“}Exx\text{”}$ ¹¹
- (Ex) $w \models_R \text{“}x \prec xx\text{”} \leftrightarrow R(\text{“}xx\text{”}, x) \wedge w \models_R \text{“}Ex\text{”}$

Call this the *Boolos model* of K . In Boolos models, plural variables are assigned many things. To be among those things is simply to be one of the assigned things. In these models, pluralities are thus nothing over and above the individual things they comprise. Boolos models, in other words, conform to the fundamental conception.

Finally, it is straightforward to prove that every Kripke model satisfying the four constraints together with one of Everything, Nothing, or Existence is isomorphic to the Boolos

¹⁰See Smith [2010].

¹¹I use Exx as an abbreviation for $\exists yy(xx = yy)$ and similarly for Ex .

model of its first-order part (theorem 1). That completes the argument.

Model-theoretically, then, the fundamental conception breaks down into four components: Downward Dependence, Upward Dependence, Extensionality, and Comprehension. More precisely, given one of Everything, Nothing, or Existence, a model conforms to the fundamental conception just in case it satisfies those components.

Let me finish this section by noting an important feature of this explication: any models with the same first-order part that conform to the fundamental conception are isomorphic. More precisely, any models that (1) satisfy the constraints—together with one of Everything, Nothing, or Existence—and (2) have the same first-order part are isomorphic (theorem 2). This means that on the model-theoretic explication, the fundamental conception provides a complete account of plural reality: keeping first-order reality fixed, any claim is either true in all of its models or false in all of them.

1.2 An axiomatic characterisation

I'll now look at the extent to which the fundamental conception can be axiomatised. I'll start by finding object language correlates of the model-theoretic constraints from the last section.

First, the successes. It is easy to see that Downward Dependence can be axiomatised by the following principle.

$$\text{(rigidity)} \quad x \prec xx \rightarrow \Box(Exx \rightarrow Ex \wedge x \prec xx)$$

More precisely, it is straightforward to show that a model satisfies Downward Dependence precisely when it validates rigidity.

And it turns out that given rigidity and a single instance of Comprehension, Upward Dependence and Extensionality can be jointly axiomatised by the following strong extensionality principle which says that pluralities comprising the same things across worlds are identical.

$$\text{(ext)} \quad \Box\forall x(\Diamond(x \prec xx) \leftrightarrow \Diamond(x \prec yy)) \rightarrow xx = yy$$

More precisely, we can show that when a model validates:

$$\exists xx\forall x(x \prec xx \leftrightarrow \Diamond(x \prec yy))$$

it satisfies Downward Dependence, Upward Dependence, and Extensionality precisely when it validates rigidity and ext (theorem 3).¹²

¹²Other standard extensionality principles fail to capture Upward Dependence in this way. Clearly, a model satisfies Extensionality precisely when it validates:

$$\text{(*)} \quad \forall xx, yy(\forall x(x \prec xx \leftrightarrow x \prec yy) \rightarrow xx = yy)$$

But there are models of Comprehension, rigidity and (*) that do not model ext and thus in which Upward Dependence fails. Indeed, there are models of Comprehension, rigidity and both of the following stronger extensionality principles that do not model ext (see Roberts [2019] footnote 78).

That leaves **Comprehension**. Unfortunately, it is well-known that **Comprehension** cannot be axiomatised, since any reasonable theory has models in which it fails.¹³ Nonetheless, it does admit of an *approximation* in the form of a comprehension schema which says of each condition φ in a fixed language that there are some things comprising the φ s. Formally:

$$\text{(comp)} \quad \exists xx \forall x (x \prec xx \leftrightarrow \varphi)$$

As we move to richer and richer languages, **comp** approaches **Comprehension** closer and closer even though it is never reached.

For our purposes, the difference between **Comprehension** and **comp** is not idle. It is natural to see **Comprehension** as modelling a certain *combinatorial* aspect of pluralities: the idea that any combination of objects gives rise to a plurality.¹⁴ The schema **comp** delivers such pluralities when the relevant combination of objects is circumscribed by a condition in some language. But there seem to be combinations of objects that may not be circumscribed by any condition in any language. It is plausible, for example, that there is a combination of set-theoretic ordered pairs which pairs each set with one and only one of its elements (Pollard [1988]). So there should be a plurality of such ordered pairs. But there is no reason to think that any condition in any language will circumscribe such a plurality. If this is right, then there will be claims—like those asserting the existence of such a ‘choice plurality’—that follow from the fundamental conception but are not provable from **rigidity + ext + comp**.¹⁵

Where does this leave us? I have shown how to axiomatise three core components of the fundamental conception—those modelled by **Downward Dependence**, **Upward Dependence**, and **Extensionality**—and I have provided an approximation for the other—namely, the combinatorial idea modelled by **Comprehension**. Clearly, the motivation for those constraints extends to show that **rigidity + ext + comp** follow from the fundamental conception. Any claim provable from them will therefore also follow from the fundamental conception. What the above example shows is that the converse may fail: there may be claims that follow from the fundamental conception which are not provable from **rigidity + ext + comp**. Nevertheless, there is a precise sense in which such claims only concern the mentioned combinatorial aspect of pluralities: it is an immediate consequence of the previous results that a model conforms to the fundamental conception just in case it validates **rigidity + ext + comp** and satisfies **Comprehension**, assuming one of **Everything**, **Nothing**, or **Existence**. Those latter claims moreover have straightforward axiomatisations. In particular, it is easy to see that a model satisfies **Nothing** precisely when it validates:

$$(**) \quad \Box \forall x (x \prec xx \leftrightarrow x \prec yy) \rightarrow xx = yy$$

$$(***) \quad \Box \forall x \Box (x \prec xx \leftrightarrow x \prec yy) \rightarrow xx = yy$$

¹³More precisely, if **Comprehension** holds in every model of \mathbb{T} , then \mathbb{T} proves, for some n , that there are necessarily n things.

¹⁴Compare with the combinatorial conception of set (Bernays [1935])

¹⁵It is a standard result in set theory that the existence of ‘choice pluralities’ is not provable from **comp** and the extensionality principle (*) from footnote 13. It follows immediately that their existence is also not provable from **rigidity + ext + comp**.

$$(no) \quad x \prec xx \rightarrow Exx$$

that it satisfies **Everything** precisely when it validates:

$$(ev) \quad x \prec xx \rightarrow \Box(\neg Exx \rightarrow x \prec xx)$$

and that it satisfies **Existence** precisely when it validates:

$$(ex) \quad x \prec xx \rightarrow \Box(\neg Exx \rightarrow [x \prec xx \leftrightarrow Ex])$$

Thus, assuming that a model validates one of **no**, **ev**, or **ex**, it conforms to the fundamental conception just in case it validates **rigidity + ext + comp** and satisfies **Comprehension**. So, speaking informally, any claim that follows from the fundamental conception is provable from **rigidity + ext + comp** when we enhance them with the combinatorial idea behind **Comprehension** (assuming one of **no**, **ev**, or **ex**). In other words, we can fully capture the fundamental conception partly precisely and partly informally. Precisely in terms of **rigidity**, **ext**, and **comp** and informally in terms of the combinatorial idea behind **Comprehension**.

It is consequence of the failure of **comp** to fully axiomatise the combinatorial idea behind **Comprehension** that **rigidity + ext + comp** fail to provide a complete account of plural reality: they have non-isomorphic models with the same first-order part. In spite of this, they do come surprisingly close. In particular, suppose we take two models K and K' of **rigidity + ext + comp** together with one of **no**, **ev**, or **ex** and assume that **comp** holds in both models for pluralities in the other model. That is, we assume that whenever X is a plurality in K , there is some Y in K' for which:

$$\bigcup_{w \in W} I_w(X) = \bigcup_{w' \in W'} I_{w'}(Y)$$

and vice versa. Then we can show that if K and K' have the same first-order part, they are isomorphic (theorem 4).

Here's another way to put the point. Suppose two speakers adopt **comp** in an *open-ended* way—so that they intend it to hold no matter which language they consider—then if they agree on first-order reality and accept **rigidity + ext + comp** (together with one of **no**, **ev**, **ex**), they will agree on plural reality.¹⁶

2 Discussion and applications

The model-theoretic and axiomatic explications of the fundamental conception show that it can be a simple, strong, and explanatory account of pluralities. As such, it can be used to

¹⁶Compare with the ‘internal’ categoricity results of Parsons [1990]. See Walsh and Button [2018] for discussion.

justify the claims that follow from it.^{17,18} This is significant because much work on the modal logic of plurals has focused on trying to justify principles like rigidity by deriving them from other claims that are, as Linnebo [2016] points out, no more clearly justified.¹⁹

The explications moreover show just how far that justification extends. To see this, it will help to compare the axioms **rigidity** + **ext** + **comp** with what I take to be the most comprehensive investigation of the principles of plural modal logic to date, namely Linnebo [2016].²⁰ In addition to **comp**, Linnebo proposes the following.²¹

$$\text{(Idsc)} \quad \forall xx, yy (\forall x (x \prec xx \leftrightarrow x \prec yy) \rightarrow \varphi(xx) \leftrightarrow \varphi(yy))$$

$$\text{(Rigid}^+\text{)} \quad \forall xx, x (x \prec xx \rightarrow \Box (Exx \wedge Ex \rightarrow x \prec xx))$$

$$\text{(Rigid}^-\text{)} \quad \forall xx, x (x \not\prec xx \rightarrow \Box (Exx \wedge Ex \rightarrow x \not\prec xx))$$

$$\text{(Dep)} \quad \forall xx, x (x \prec xx \rightarrow \Box (Exx \rightarrow Ex))$$

$$\text{(BC)} \quad x \prec xx \rightarrow Ex^{22,23}$$

Although extensive, there are important claims that these principles fail to imply.

Recall our people—Melesha, Nadia, and Dylan—and consider them without Melesha. To avoid confusion, let xx be the former people and yy the latter. Are the yy among the xx ,

¹⁷In this respect, the fundamental conception is similar to the so-called *iterative conception of set* in set theory. See Boolos [1971] and Maddy [1988].

¹⁸Roberts [2016], Roberts [2019], and Linnebo [2016] also suggest using the fundamental conception to justify claims in the language of plural modal logic.

¹⁹See, for example, Uzquiano [2011] and Rumfitt [2005].

²⁰It is important to keep in mind that Linnebo [2016] defines the existence predicate for pluralities, Exx , as $\exists x (x \prec xx)$. This will not effect the claims I make below, but it is worth noting that once we define Exx in this way, what are usually logical and innocuous principles can become substantial. For example, instances of the axiom of universal instantiation— $Exx \wedge \forall yy \varphi \rightarrow \varphi(xx)$ —becomes non-trivial.

²¹Linnebo [2016] actually gives some reasons for denying **comp** in full generality (see the discussion in footnote ? for the impact of this on the current proposal). That won't effect what I will say, though.

²²This is the first-order “being constraint” for \prec . See Williamson [2013]. The plural being constraint is no. Given Linnebo’s definition of “ Exx ”, no follows from BC.

²³Linnebo also suggests the following “traversability principle”:

$$\text{(UnivTrav-C)} \quad \forall xx \Box (Exx \rightarrow \forall x (x \prec xx \leftrightarrow \bigvee_{a \prec xx} x = a))$$

where “ $\bigvee_{a \prec xx} x = a$ ” is intended to express an infinite disjunction of claims of the form “ $x = a$ ” where a is one of the xxs . But this is redundant. It is easy to see that **Rigid**⁺, **Dep**, and **BC** already imply **rigidity** and that **rigidity** in turn ensures **UnivTrav-C** on its intended reading. Though see footnote 29 for a criticism of the formalism used in this principle.

whenever the xx exist? Do the yy exist whenever the xx do? The fundamental conception implies positive answers to both questions. If these and those are nothing over and above the individual things they comprise and these are among those, then whenever those exist these must also exist and be among them. Formally:

$$\text{(sub-plurality)} \quad \forall xx, yy (xx \subseteq yy \rightarrow \Box(Eyy \rightarrow Exx \wedge xx \subseteq yy))$$

We can show that sub-plurality is provable from rigidity + ext + comp.²⁴ And it has important applications in the philosophy of mathematics. Take *modal structuralism*: the view that mathematics concerns possible structures. On the most prominent version of this view, due to Geoffrey Hellman, structures are pluralities.²⁵ It turns out that for such structures to be well-behaved—in particular, for a structure to satisfy the same second-order claims wherever it exists—sub-pluralities of a structure must continue to exist and be its sub-pluralities wherever that structure exists. This is precisely what sub-plurality ensures ([Roberts, 2019, p. 18]).

But sub-plurality does not follow from Linnebo’s principles.²⁶ More generally, it is easy to see that sub-plurality is subsumed by Upward Dependence and that it is this aspect of pluralities that Linnebo’s principles fail to address. Indeed, there are simple Kripke models of those principles in which the first-order domains are constant but in which the plural domains are completely disjoint.^{27,28}

Let me end with two more applications of the idea behind Upward Dependence.

Perhaps the most well-known use of pluralities in the philosophy of mathematics is in securing categoricity results. It is often claimed, for example, that plural formulations of arithmetic have exactly one model up to isomorphism: in so far as there are non-standard models, they can be rejected as illegitimate. More generally, plural quantification is taken to be *determinate*: given any first-order domain, there is only one legitimate corresponding plural domain. Florio and Linnebo [2016] have challenged this orthodoxy. They show that there are non-standard *plurality-based* models—where the domain of the plural quantifiers consists of pluralities—and argue that there is no reason to dismiss such models as illegitimate. More precisely, they argue that there is a symmetry between plural and set quantification: extant reasons to dismiss non-standard models of plural quantification are also reasons to dismiss non-standard models of set quantification. The detour through pluralities in order to secure

²⁴See Roberts [2019] footnote 78 for the proof.

²⁵See Hellman [1989] and Roberts [2019] for details.

²⁶See again Roberts [2019] footnote 78 for the proof.

²⁷To see this, take any Kripke model with a constant first-order domain in which the plural domains are disjoint copies of its powerset. If Y is the copy of X at w , then we let $I_w(Y) = X$. That will already satisfy all of Linnebo’s principles with the exception of BC. To make sure the model also satisfies BC we just require that $x \prec X$ is trivially false at worlds at which X does not exist. Formally, we require that when $X \notin D_2(w)$, $w \models x \not\prec X$. In other words, we adopt **Nothing**.

²⁸In a footnote, Linnebo also suggests the following principle as an alternative to Dep:

$$\forall xx \Box (Exx \leftrightarrow \bigwedge_{a \prec xx} Ea)$$

Once this is added, it turns out that on a natural model theory for “ $\bigwedge_{a \prec xx}$ ”, we do get Upward Dependence. The problem is that although this new resource is tolerably clear in the model theory, it is completely unclear how to axiomatise it.

categoricity would thus be redundant. I will now argue that the fundamental conception, and in particular Upward Dependence, can be used to break this symmetry.

It will help to start with an argument for the determinacy of plural quantification that Florio and Linnebo consider and reject. The argument, which they attribute to Hossack [2000], is based on the claim that plural quantification is *ontologically innocent*. On one natural way of making this out, it says that the pluralities in the range of plural quantifiers do not incur ontological commitment over and above the ontological commitments of the first-order quantifiers. Ontological innocence would then be determined *point-wise*, plurality by plurality: if each plurality in the domain of the plural quantifiers is nothing over and above some individual things in the domain of the first-order quantifiers, then the plural quantifiers are ontologically innocent. This is clearly in keeping with the fundamental conception. But as Florio and Linnebo point out, it is not enough to ensure determinacy. Any plurality-based model respects ontological innocence in this sense and since Florio and Linnebo have shown that there are non-standard plurality-based models, ontological innocence in this sense does not ensure determinacy.

This is certainly right. But there are other equally natural ways to understand ontological innocence. Say that a plural quantifier is ontologically innocent if the pluralities in its range conform to the fundamental conception. On this sense, ontological innocence is determined *globally*: if the pluralities in the range of the plural quantifiers respect the fundamental conception as a whole, then the plural quantifiers are ontologically innocent. Once we modify ontological innocence in this way, however, we can rule out Florio and Linnebo’s non-standard plurality-based models. In particular, we can show that any two models with the same first-order domain that conform to the fundamental conception agree on their pluralities. The argument is simple and relies crucially on Upward Dependence.

Florio and Linnebo use the plural predicate variables $\mathbf{D}, \mathbf{D}', \mathbf{D}'', \dots$ etc to circumscribe plural domains. Now suppose we have two models \mathcal{M} and \mathcal{M}' with the same first-order domain D and respective plural domains \mathbf{D} and \mathbf{D}' . Suppose furthermore that their plural domains differ. Say $\mathbf{D}(xx)$ but not $\mathbf{D}'(xx)$. By Downward Dependence applied to \mathcal{M} , each of the xx s is in the domain D . Thus, by Upward Dependence applied to \mathcal{M}' , the xx s must exist according to \mathcal{M}' , which is to say $\mathbf{D}'(xx)$. Contradiction!

In short, pluralities are so closely tied to the things they comprise that they aren’t an extra ontological commitment beyond those things. But they’re also so closely tied to the things they comprise that they can’t help but exist when the things they comprise exist. For pluralities, ontological innocence cuts both ways. Florio and Linnebo’s non-standard plural-based models can thus be ruled out as illegitimate. In general, if plural quantifiers are ontologically innocent in the sense that the pluralities in their ranges conform to the fundamental conception, then plural quantification is determinate.

Williamson [2013] is a sustained defence of *necessitism*—the view that existence is necessary—against *contingentism*—the view that it isn’t. One part of that defence concerns the kind of plural modal logic that each view allows for. He argues, in particular, that the contingentist lacks an adequate way to express very natural principles. For example, it looks like the contingentist cannot formulate a “plural analogue of identity” or a principle that ensures instances of Upward Dependence like:

$$(*) \quad \forall xx, x(\forall y(y \prec xx \leftrightarrow y = x) \rightarrow \Box(Ex \rightarrow Exx))$$

In order to do so, Williamson claims, the contingentist would have to avail themselves of so-called *backtracking operators* ([Williamson, 2013, p. 250-252]). Such operators effectively allow worlds to be cross referenced: to be able to refer in one world back to what is true in another. The worry is that these operators are really just hidden quantifiers over possible worlds.²⁹ But as Williamson argues, it is unclear whether the contingentist can countenance such quantification. For the contingentist, modal operators should be primitive, irreducible to quantification.

What my axiomatisation of the fundamental conception shows is that the contingentist can both formulate a notion of identity for pluralities and justify the relevant instances of Upward Dependence.³⁰ In particular, the extensionality principle *ext* shows that “ $\Box\forall x(\Diamond(x \prec xx) \leftrightarrow \Diamond(x \prec yy))$ ” can be treated as a plural analogue of identity without the need for backtracking operators. More precisely, although I have stated *ext* as a principle concerning a primitive relation of identity for pluralities, it could just as well be replaced by a principle which says that this *is* an identity relation. Formally:

$$(**) \quad \Box\forall x(\Diamond(x \prec xx) \leftrightarrow \Diamond(x \prec yy)) \rightarrow \varphi(xx) \leftrightarrow \varphi(yy)$$

The theory *rigidity* + *ext* + *comp* with the identity relation holding between pluralities is then equivalent to *rigidity* + **** + *comp* with identity confined to first-order terms.³¹ Furthermore, the proof that *sub-plurality* follows from *rigidity* + *ext* + *comp* extends to show (*) and other similar claims also follow.³²

3 Technical appendix

In this appendix, I provide proofs for the main results of the paper. Throughout I will assume that we have uniformly settled on adopting Existence, Ex, or ex. All the proofs I give can be easily modified for the other principles.

Theorem 1. *Let K be a model of Downward Dependence, Upward Dependence, Extensionality, Comprehension, and Existence. Then K is isomorphic to the Boolos model of its first-order part.*

²⁹See Melia [1992].

³⁰The Kripke models I’ve used have variable domains and all the proof theoretic results can be carried out in a simple positive free version of S5 modal logic.

³¹Technically, they are bi-interpretable.

³²Williamson also claims that the necessitist has a simpler account of pluralities than the contingentist. It is certainly true that for the necessitist, *rigidity* and *ext* have slightly simpler formulations. In particular, *rigidity* is equivalent to:

$$x \prec xx \rightarrow \Box x \prec xx$$

for the necessitist, and *ext* is equivalent to:

$$\forall x(x \prec xx \leftrightarrow x \prec yy) \rightarrow xx = yy$$

But if the necessitist and the contingentist both motivate their respective principles using the fundamental conception, then they have precisely the same account of pluralities: namely, the fundamental conception. As I see it, the crucial difference between the contingentist and the necessitist with respect to plurals comes down to which of *ev*, *no*, and *ex* they adopt. For the necessitist, they all follow from *rigidity*. The contingentist, on the other hand, has a genuine choice, and it is unclear what principled reasons they could give for choosing one over the others.

Proof. Let K be as in the theorem and let K' be its first-order part. Let a be a variable assignment over K and let A_a be the following Boolos assignment:

$$\langle \text{"}x\text{"}, y \rangle \in A_a \leftrightarrow a(\text{"}x\text{"}) = y$$

$$\langle \text{"}xx\text{"}, y \rangle \in A_a \leftrightarrow y \in \bigcup_{w \in W} I_w(a(\text{"}xx\text{"}))$$

By a simple induction on the complexity of formulas we show that:

$$w \vDash_a \varphi \leftrightarrow w \vDash_{A_a} \varphi$$

The only non-trivial cases are those for " $x \prec xx$ ", " $xx = yy$ ", and " $\exists xx\varphi$ ". For " $x \prec xx$ ", first note that by Ex, $w \vDash_{A_a} \text{"}x \prec xx\text{"}$ iff $\langle \text{"}xx\text{"}, a(\text{"}x\text{"}) \rangle \in A_a$ and $a(\text{"}x\text{"}) \in D_1(w)$. By the definition of A_a , that is in turn equivalent to $a(\text{"}x\text{"}) \in \bigcup_{w \in W} I_w(a(\text{"}xx\text{"})) \cap D_1(w)$. Now if $a(\text{"}xx\text{"}) \in D_2(w)$, then $I_w(a(\text{"}xx\text{"})) = \bigcup_{w \in W} I_w(a(\text{"}xx\text{"})) \cap D_1(w)$ by Downward Dependence; but also if $a(\text{"}xx\text{"}) \notin D_2(w)$, then $I_w(a(\text{"}xx\text{"})) = \bigcup_{w \in W} I_w(a(\text{"}xx\text{"})) \cap D_1(w)$ by Existence. So, in either case, $w \vDash_{A_a} \text{"}x \prec xx\text{"}$ iff $w \vDash_a \text{"}x \prec xx\text{"}$.

For " $xx = yy$ ", first note that if $a(\text{"}xx\text{"}) = a(\text{"}yy\text{"})$, then $\bigcup_{w \in W} I_w(a(\text{"}xx\text{"})) = \bigcup_{w \in W} I_w(a(\text{"}yy\text{"}))$ and thus $w \vDash_{A_a} \text{"}xx = yy\text{"}$. Conversely, suppose $w \vDash_{A_a} \text{"}xx = yy\text{"}$ and thus $\bigcup_{w \in W} I_w(a(\text{"}xx\text{"})) = \bigcup_{w \in W} I_w(a(\text{"}yy\text{"}))$. Now, let w' be such that $a(\text{"}xx\text{"}) \in D_2(w')$. Then, by Downward Dependence, $\bigcup_{w \in W} I_w(a(\text{"}xx\text{"})) = \bigcup_{w \in W} I_w(a(\text{"}yy\text{"})) \subseteq D_1(w')$ and so by Upward Dependence $a(\text{"}yy\text{"}) \in D_2(w')$. It follows from Extensionality and Downward Dependence that $a(\text{"}xx\text{"}) = a(\text{"}yy\text{"})$ as required.

For " $\exists xx\varphi$ ", assume that $w \vDash_a \text{"}\exists xx\varphi\text{"}$. Then $w \vDash_{a'} \varphi$ for some a' that differs from a at most in what it assigns to " xx " where that thing exists at w : that is, $a'(\text{"}x\text{"}) \in D_1(w)$. Thus $w \vDash_{A_{a'}} \varphi$ by the induction hypothesis and $\bigcup_{w \in W} I_w(a(\text{"}xx\text{"})) \subseteq D_1(w)$ by the definition of $A_{a'}$. So, $w \vDash_{A_a} \text{"}\exists xx\varphi\text{"}$. Conversely, suppose $w \vDash_{A_a} \text{"}\exists xx\varphi\text{"}$. Then $w \vDash_{A'} \varphi$ for some A' that differs from A_a at most in what it relates to " xx " where those things all exist at w : that is, $\{x : A'(\text{"}xx\text{"}, x)\} \subseteq D_1(w)$. Now, by Comprehension, there will be an $X \in D_2(w)$ for which $I_w(X) \cap D_1(w) = \{x : A'(\text{"}xx\text{"}, x)\}$. By Downward Dependence, $I_w(X) \cap D_1(w) = I_w(X) = \bigcup_{w \in W} I_w(X)$. Thus, where a' is an assignment just like a expect possibly in that it assigns X to " xx ", $A' = A_{a'}$. It follows that $w \vDash_a \text{"}\exists xx\varphi\text{"}$. \square

Theorem 2. *Let K and K' be models of Downward Dependence, Upward Dependence, Extensionality, Comprehension, and Existence. Assume, moreover, that they have the same first-order part. Then, they are isomorphic.*

Proof. Let X be a plurality in K and assume that it exists at w . By Downward Dependence, $I_w(X) = \bigcup_{w \in W} I_w(X) \subseteq D_1(w)$. By Comprehension, there is some Y in K' that exists at w for which $I'_w(Y) \cap D_1(w) = \bigcup_{w \in W} I_w(X)$. Since Y exists at w it follows from Downward Dependence that $I'_w(Y) \cap D_1(w) = \bigcup_{w \in W} I'_w(Y)$ and thus that $\bigcup_{w \in W} I'_w(Y) = \bigcup_{w \in W} I_w(X)$. By Downward Dependence, Upward Dependence, and Extensionality, there is at most one plurality comprising a given set of things: that is, for Z, Z' in a given model of Downward Dependence, Upward Dependence, and Extensionality, if $\bigcup_{w \in W} I_w(Z) = \bigcup_{w \in W} I_w(Z')$, then $Z = Z'$. (This is precisely what ext says.) So, Y is unique. In general, for X in K , let $f(X)$ be the unique such Y .

By induction on the complexity of formulas we prove that:

$$w \models_K \varphi(\vec{X}) \leftrightarrow w \models_{K'} \varphi(f(\vec{X}))$$

For “ $x \prec xx$ ”: if $X \in D_2(w)$, then $f(X) \in D'_2(w)$ and $I_w(X) = \bigcup_{w \in W} I_w(X) = \bigcup_{w \in W} I'_w(Y) = I'_w(Y)$ by Upward Dependence and Downward Dependence. Thus, $w \models_K “x \prec X”$ iff $w \models_{K'} “x \prec X”$. If $X \notin D_2(w)$, then $Y \notin D'_2(w)$ and $I_w(X) = \bigcup_{w \in W} I_w(X) \cap D_1(w) = \bigcup_{w \in W} I'_w(Y) \cap D_1(w) = I'_w(Y)$ by Downward Dependence, Upward Dependence, and Existence. Thus, again, $w \models_K “x \prec X”$ iff $w \models_{K'} “x \prec X”$.

By Downward Dependence, Upward Dependence, and Extensionality, $w \models “X = Y”$ iff $X = Y$ iff $\bigcup_{w \in W} I_w(X) = \bigcup_{w \in W} I_w(Y)$ which in turn is equivalent to $\bigcup_{w \in W} I'_w(f(X)) = \bigcup_{w \in W} I'_w(f(Y))$ and thus to $f(X) = f(Y)$ and to $w \models “f(X) = f(Y)”$. In other words, f is one-one.

To complete the proof, we just need to show that f onto. So let Y be a plurality in K' and suppose that $Y \in D'_2(w)$. Then by Downward Dependence $I'_w(Y) = \bigcup_{w \in W} I'_w(Y) \subseteq D_1(w)$. So, by Comprehension, there is some $X \in D_2(w)$ such that $I_w(X) \cap D_1(w) = I'_w(Y)$. Thus, by Downward Dependence, $\bigcup_{w \in W} I_w(X) = I_w(X) = I_w(X) \cap D_1(w) = I'_w(Y) = \bigcup_{w \in W} I'_w(Y)$. So, $f(X) = Y$. \square

Theorem 3. *Suppose that K models rigidity and the following instance of comp:*

$$\exists x x \forall x (x \prec xx \leftrightarrow \diamond(x \prec yy))$$

Then, K models ext just in case it satisfies Extensionality and Upward Dependence.

Proof. Let K be as in the theorem, and assume that it models ext. Let $X, Y \in D_2(w)$ be such that $I_w(X) \cap D_1(w) = I_w(Y) \cap D_1(w)$. By rigidity, $\bigcup_{w \in W} I_w(X) = I_w(X) = I_w(X) \cap D_1(w) = I_w(Y) \cap D_1(w) = I_w(Y) = \bigcup_{w \in W} I_w(Y)$. So, by ext, $X = Y$. In other words, K satisfies Extensionality.

Now assume that $\bigcup_{w \in W} I_w(X) \subseteq D_1(w)$. By the above instance of comp there is some $Y \in D_2(w)$ for which $I_w(Y) \cap D_1(w) = \bigcup_{w \in W} I_w(X)$. Now, since $Y \in D_2(w)$, $I_w(Y) \cap D_1(w) = I_w(Y) = \bigcup_{w \in W} I_w(Y)$. Thus, $\bigcup_{w \in W} I_w(X) = \bigcup_{w \in W} I_w(Y)$. By ext it follows that $X = Y$. In other words, K satisfies Upward Dependence.

Finally, and conversely, assume that K satisfies Extensionality and Upward Dependence. Then since it models rigidity, it also satisfies Downward Dependence. It follows from all three principles that $X = Y$ iff $\bigcup_{w \in W} I_w(X) = \bigcup_{w \in W} I'_w(Y)$. \square

Theorem 4. *Let K and K' be two Kripke models with the same first-order part such that for $X \in D_2(w)$, there is $Y \in D'_2(w)$ for which:*

$$I_w(X) \cap D_1(w) = I'_w(Y) \cap D_1(w)$$

and vice versa. Assume, moreover, that they both model comp + rigidity + ext + ex. Then, they are isomorphic.

Proof. By the previous theorem, we know that K and K' both satisfy Upward Dependence, Downward Dependence, and Extensionality. Moreover, ex is valid just in case Existence holds. Finally, note that the proof of theorem 2 only relied on the assumption about comprehension in the statement of this theorem. \square

References

- P. Bernays. On Platonism in Mathematics. In *Philosophy of Mathematics: Selected Readings*. Cambridge University Press, 1935.
- G. Boolos. The iterative conception of set. *Journal of Philosophy*, 68(8):215–231, 1971.
- G. Boolos. Nominalist platonism. *The Philosophical Review*, 94(3):327–344, 1985. ISSN 00318108, 15581470. URL <http://www.jstor.org/stable/2185003>.
- S. Florio and O. Linnebo. On the innocence and determinacy of plural quantification. *Nous*, 50(3):565–583, 2016. doi: 10.1111/nous.12091.
- G. Hellman. *Mathematics without Numbers: Towards a Modal-Structural Interpretation*. Clarendon, Oxford, 1989.
- K. Hossack. Plurals and complexes. *British Journal for the Philosophy of Science*, 51(3): 411–443, 2000. doi: 10.1093/bjps/51.3.411.
- O. Linnebo. Plurals and modals. *Canadian Journal of Philosophy*, 46(4-5):654–676, 2016. doi: 10.1080/00455091.2015.1132975.
- . Linnebo. Plural quantification. In E. N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, summer 2017 edition, 2017.
- P. Maddy. Believing the Axioms. I. *The Journal of Symbolic Logic*, 53(2):481–511, 1988.
- J. Melia. Against modalism. *Philosophical Studies*, 68(1):35–56, 1992. doi: 10.1007/BF00354468.
- C. Parsons. The uniqueness of the natural numbers. *Iyyun: The Jerusalem Philosophical Quarterly*, 39:13–44, 1990.
- S. Pollard. Plural quantification and the axiom of choice. *Philosophical Studies: An International Journal for Philosophy in the Analytic Tradition*, 54(3):393–397, 1988. ISSN 00318116, 15730883. URL <http://www.jstor.org/stable/4319993>.
- S. Roberts. *Potentialism and reflection*. PhD thesis, Birkbeck, University of London, London, 2016.
- S. Roberts. Modal structuralism and reflection. *The Review of Symbolic Logic*, 12(4):823860, 2019. doi: 10.1017/S1755020318000138.
- I. Rumfitt. Plural terms : Another variety of reference? In J. L. Bermúdez, editor, *Thought, Reference, and Experience: Themes From the Philosophy of Gareth Evans*, pages 84–123. Clarendon Press, 2005.
- P. Smith. Squeezing arguments. *Analysis*, 71(1):22–30, 12 2010. ISSN 0003-2638. doi: 10.1093/analys/anq120. URL <https://doi.org/10.1093/analys/anq120>.
- G. Uzquiano. Plural quantification and modality. *Proceedings of the Aristotelian Society*, 111 (2pt2):219–250, 2011.

- S. Walsh and T. Button. *Philosophy and Model Theory*. Oxford, UK: Oxford University Press, 2018.
- T. Williamson. *Modal Logic as Metaphysics*. Oup Oxford, 2013.
- S. Yablo. Circularity and paradox. In T. Bolander, V. F. Hendricks, and S. A. Pedersen, editors, *Self-Reference*, pages 139–157. CSLI Publications, 2006.