Ultimate V

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Potentialism

Potentialism is the view that the universe of mathematics is in some sense inherently potential. It comes in two main flavours.

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Take some people: Nadia, Dylan, and Melesha. Since each of them exists, the height potentialist claims that there could have been a set of them: the set {Nadia, Dylan, Melesha} could have existed. Once we have that set, we can repeat the process. Taking each of Nadia, Dylan, and Melesha together with the new set, the height potentialist will claim that *they* could have formed a set: the set {Nadia, Dylan, Melesha, {Nadia, Dylan, Melesha}} could have existed.

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Continuing in this way, we get the possibility of more and more sets. So many, according to the height potentialist, that the sets obtained in this way satisfy the axioms of set theory.

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According to the width potentialist, there is thus no universe containing absolutely all subsets of the natural numbers and so no universe containing absolutely all sets simpliciter. No universe of sets is privileged on this account: there are many universes, containing different sets, and making different claims true. There is no ultimate background universe of sets, no ultimate V.

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I will argue in this talk that they aren't. **Height and width potentialism are inconsistent with one another**.

In particular, I will argue that the possible sets according to the height potentialist constitute an ultimate universe of sets, an ultimate V: a universe from which we cannot apply the method of forcing to obtain new universes of sets.

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- I'll then show that given plausible background assumptions, they are inconsistent with one another.
- I'll end by considering some responses.

Motivating height potentialism

Height potentialism is motivated by the paradoxes.

Plural Russell's paradox

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First, there's the plural comprehension schema, which says that any condition determines a plurality: for any condition ϕ, there are some things which comprise all and only the ϕs. Formally:

(plural comp) $$\exists xx \forall x(x \prec xx \leftrightarrow \phi)$$

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Second, there is a principle which says that pluralities *collapse* to sets: that any things whatsoever form a set. Formally:

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So, which assumption should we reject?

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The ϕs are thus nothing over and above the individual things that happen to be ϕ. Since each individually ϕ exists, nothing more is needed for there to be some things comprising all and only the ϕs.

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It looks like **plural comp** is clearly true.

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Their central idea is to solve the paradox by claiming that although these arguments *are* compelling, rather than justifying **collapse**, they justify a similar but importantly weaker claim: namely, the claim that any things *could* have formed a set. Formally:

(collapse$^{\Diamond}$) $$\Box\forall xx\Diamond\exists x(x \equiv xx)$$

This modal version of collapse is, unlike **collapse** itself, perfectly consistent with **plural comp**.

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Although this is a crucial question for the height potentialist, I will ignore it in what follows. All authors agree that the modal logic governing $\Diamond$ should be S4.2 plural modal logic together with suitable assumptions about the modal behaviour of pluralities and sets. This will suffice for the results I prove.

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What collapse$^\Diamond$ says, then, is that any possible plurality is collectable.

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The crucial claim is that the opponent of collapse$^{\Diamond}$ cannot meet this explanatory challenge in a satisfactory way.

Actualism

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For some things to possibly form a set is then for them to actually form a set, on this account, and collapse$^\Diamond$ becomes equivalent to collapse. Thus, since the non-self-membered sets don't actually form a set, they couldn't have formed a set and collapse$^\Diamond$ is false.

Actualism

In general, the account implies that there is no difference between possible existence and actual existence so that when we restrict our attention to claims solely about sets and pluralities, the modality becomes redundant. Call this *actualism*. Formally:

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The reason given is that neither of the two standard ways for the actualist to meet the challenge—using the *limitation of size* or *iterative* conceptions of set—provide a satisfactory response.

Limitation of size

According to the limitation of size conception of sets, some things form a set precisely when they are fewer than the ordinals. The ordinals thus provide a threshold cardinality below which pluralities form sets and above which, they don't. On this view, the most natural response to challenge is to claim that what makes the uncollectable pluralities uncollectable is that they are "too large": that they are not fewer than the ordinals.

Iterative conception of set

According to the iterative conception of sets, the sets occur in a well-ordered series of stages. At the very first stage, we have no sets whatsoever. Then, at the second stage, we have all the sets of things at the first stage: that is, since there is nothing at the first stage, we have the empty set! At the third stage, we have all the sets of things at the second stage: that is, since the empty set is the only thing at the second stage, we have precisely the set containing the empty set and the empty set itself. At the fourth stage, we have all the sets of *those* things. And so on indefinitely. In general, at any stage we have sets of any things which all occur together at some previous stage. On this view, some things form a set just in case they all occur together at some stage and so the most natural response to the challenge is to claim that what makes the uncollectable pluralities uncollectable is that there is no stage at which the things among them all together.

The charge is that each of these responses fails in important cases. For example, we know that the ordinals do not form a set. According to the limitation of size response:

> *the explanation is that [the ordinals] are too many to form a set, where being too many is defined as being as many as [the ordinals]. Thus, the proposed explanation moves in a tiny circle. (Linnebo, p. 154, Pluralities and sets.)*

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The actualist faces an explanatory challenge that they fail to meet. The height potentialist faces no such challenge, since they accept **collapse**$^{\Diamond}$. Other things being equal, potentialism should thus be preferred.

Beyond **collapse**$^\lozenge$

So, that's the primary argument for **collapse**$^\lozenge$.

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I'm now going to argue that these further principles are not optional: the argument for **collapse**$^{\Diamond}$ we've considered generalises to an argument for the claim that the axioms of $\mathbf{ZC} + \forall x \exists \alpha (x \in V_\alpha)$ hold in the potential sets.

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We are thus faced with *another* explanatory challenge: we are owed an explanation of what makes the uncollected pluralities uncollected in a given world.

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Their response to that challenge, we can assume, is either derived from the limitation of size or iterative conceptions and according to the height potentialist it will fail to be explanatory in certain crucial cases.

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For the height potentialist, the modality is certainly not redundant: **collapse**$^\Diamond$ is inequivalent to **collapse**—the first true, the second false. The two explanatory challenges are thus also inequivalent for them.

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And although they sidestep the challenge to explain what makes the uncolletable pluralities uncollectable—since they think there could not have been any such pluralities—they face the challenge to explain what makes the uncollected pluralities uncollected head on.

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It should be clear that an appeal at this point to either the limitation of size or iterative conceptions would the earlier argument for **collapse**$^\Diamond$ undermined. For then the proposed explanations would be precisely the same as those offered by the actualist. Each would be equally unexplanatory. The actualist would effectively face one challenge—since both are equivalent—and give an somewhat unexplanatory response and the potentialist would effectively face one challenge—since one does and one doesn't apply to them—and give an equally unexplanatory answer. A stalemate.

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It is based on the idea that the elements of a set are prior to the set: that the elements of a set must exist *before* the set can exist.

To make sense of this idea, we need another modal operator: one that expresses the "before", a dual to $\Diamond$ that "looks backïnstead of forward. Formally, we can add to our language a new pair of operators $\Diamond^{<}$ and $\Box^{<}$ meaning roughly that it will and must be the case respectively and a pair of operators $\Diamond^{>}$ and $\Box^{>}$ meaning it was and always was the case respectively. Let $\Box$ be an operator which says that it always was, is, and always will be the case. Formally, $\Box\phi$ just in case $\Box^{<}\phi \wedge \phi \wedge \Box^{>}\phi$.

Beyond **collapse**$^{\Diamond}$

The priority idea can then be expressed as follows.

(priority) $$\Box\forall x\Diamond^{>}\exists xx(x \equiv xx)$$

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But it does not give us a lower bound. **collapse**$^\Diamond$ ensures that any things will form a set at some later world, but it does not tell us when: we may have to pass through many worlds to get it.

Beyond **collapse**$^\Diamond$

Fortunately, a natural strengthening of **collapse**$^\Diamond$ does give us a lower bound. This strengthening says that some things are *sufficient* for the corresponding set: once they exist, the set *must* exist. Formally:

(plenitude) $$\Box\forall xx \Box^{<} \exists x(x \equiv xx)$$

Together, then, **priority** and **plenitude** tell us that the pluralities which are collected at a given world are precisely those whose elements exist at some prior world. Formally:

$$\Box\forall xx(\exists x(x \equiv xx) \leftrightarrow \Diamond^{<} Exx)$$

Since **plenitude** implies **collapse**$^\Diamond$, we get a response to both challenges.

Summary

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Those principles, in turn, imply that a large fragment of ZFC holds in the possible sets. As Studd shows, they imply that the axioms of ZC + $\forall x \exists \alpha (x \in V_\alpha)$ hold in the possible sets.

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We need not go into the details. What matters for us is one particular consequence of this claim, namely: that we can always add subsets to a universe.

In particular, given any universe $\mathcal{U}$ and $x \in \mathcal{U}$, there is another universe $\mathcal{U}'$ and $y \in \mathcal{U}'$ such that $y \subseteq x$ and $y \notin \mathcal{U}$. (Indeed, every non-trivial forcing will add at least one such subset (ignoring class forcing.)

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Indeed, despite significant efforts, set-theorists and philosophers have failed to find compelling *new* principles that might prove or disprove **CH**.

Width potentialism deals with this problem extremely well. According to the view, the attempt to settle such questions is misplaced. **CH** is not an unambiguous statement for which we can marshal evidence. Rather, it is true only relative to a universe of sets. And in the broad space of universes of sets, we already know how **CH** behaves: how it is true some universes and false in others. There is no ultimate V in which **CH** either unambiguously holds or fails to hold.

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(Moreover, if the height potential sets satisfy the axiom of countable replacement, then every set in any universe is in the potential sets if it's well-founded or a copy of it is, if it isn't.)

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So, x could have existed in the height potentialist's sense. Then there could have been a plurality of the things in x that are in y according to $\mathcal{U}$ by **plural comp**. Since every element of y is in x, that plurality comprises all the elements of y: it is co-extensive with y. By **collapse**$^{\Diamond}$, it could have formed a set. So, there could have been a set co-extensive with y. Since co-extensive sets are identical by extensionality, y could have existed in the height potentialist's sense.

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In addition to the height potentialist's $\Diamond$, I'll adopt two modal operators $\blacklozenge$ which will be used to represent the width potentialist sense in which we might add subsets to a given set. I will assume both operators are governed by the minimal modal logic K. I'll also adopt an actuality operator @.

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The actuality operator is intended to be used to **rigidly** refer back to the actual circumstances and will allow me to avoid unhelpful interactions between the other modal operators. I will assume it is governed by the modal logic K together with:

$$\Diamond @\phi \vee \blacklozenge @\phi \rightarrow @\phi$$

$$@\neg\phi \leftrightarrow \neg @\phi$$

Sets and pluralities are extensional

Let $\Diamond^*$ be the disjunction of both modalities prefixed with the actuality operator. That is:

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Then I will assume the following two principles, which are intended to jointly capture the idea that sets are extensional in nature.

(extensionality) $\Box^*\forall x\Box^*\forall y(\Box^*\forall z(\Diamond^* z \in x \leftrightarrow \Diamond^* z \in y) \to x = y)$

(rigidity$_s$) $x \in y \to \Box^*(Ey \to Ex \wedge x \in y)$

Since pluralities are also extensional, I will assume rigidity for them too.

(rigidity$_p$) $x \prec xx \to \Box^*(Exx \to Ex \wedge x \prec xx)$

Height potentialism

I will now formulate height potentialism within a universe.

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Let $xx, yy, zz, \ldots$ range plurally over whatever the first-order quantifiers range over. I will assume a plural comprehension schema governing them.

(plural comp) $\exists xx \forall x (x \prec xx \leftrightarrow \phi)$

I will say that $\Diamond$ is **height potential** when the following principle holds.

(collapse$^{\Diamond}$) $@\Box\forall xx @\Diamond\exists x (x \equiv xx \wedge Exx)$

where $x \equiv xx$ abbreviates $\forall y (y \in x \leftrightarrow y \prec xx)$.

Width potentialism

Recall that forcing extensions add subsets. So, if $\blacklozenge$ represents a forcing extension of $\lozenge$, then it will typically add subsets of sets which exist according to $\lozenge$. Formally:

(add subsets) $$@\lozenge\exists x @\blacklozenge\exists y(y \subseteq x \wedge \neg @\Box E y)$$

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Assume **extensionality**, **rigidity**$_s$, **rigidity**$_p$, *and* **plural comp**. *Then,* **collapse**$^\Diamond$ *is inconsistent with* **add subsets**

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In so far as we are interested in CH, any of these universes will provide an ultimate universe of sets, an ultimate V.

Possible responses

How might we resist this result?

1. Reject the main argument for height potentialism and find an alternative that motivates **collapse**$^\Diamond$ without motivating the claim that the potential sets constitute a universe of sets.
   - Although other arguments for **collapse**$^\Diamond$ have been floated—e.g. arguments from liberalism about possibility—it is unclear whether they can tread such a fine line.
   - Since the height potential powerset of ω, for example, contains absolutely all subsets of ω, it cannot exist in any universe according to the width potentialist.
   - To avoid this, we'd need a motivation for **collapse**$^\Diamond$ that doesn't motivate the powerset axiom for the potential sets.

Possible responses (II)

2. Reject one of the assumptions in the proof that we cannot add subsets to the height potential sets.
   - I think there is only one option here: reject **plural comp**.
     - Indeed, some have suggested that **plural comp** *should* be rejected for some conditions.
   - There are a number of problems with this response.
     - The instance of plural comprehension we need is:

       $$\forall x \exists xx \forall z (z \prec xx \leftrightarrow z \in x \wedge z \in y)$$

       Since every set determines a plurality, that's implied by:

       $$\forall yy \exists xx \forall z (z \prec xx \leftrightarrow z \prec yy \wedge z \in y)$$

       Which is effectively a form of plural separation. If each individual ϕ is among the *yy*s, and pluralities are nothing over and above the individual things they comprise, then it is very hard to see how we could deny that there is a plurality of the ϕs.

Possible responses (III)

- In fact, everyone who rejects **plural comp** in full generality still accepts this separation principle.
- In any case, this strategy does not sit well with the height potentialist's initial argument.
- If we give up the simple account of pluralities as nothing over and above the individual things they comprise, then we need to replace it with some other account. And it's unclear if there is an account of pluralities where plural separation fails that is more explanatory than the limitation of size or iterative conceptions according to the actualist.

Thanks!