Ultimate V

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1 Introduction

Potentialism is the view that the universe of mathematics is in some sense inherently potential. It comes in two main flavours. *Height potentialism* is based on the idea that a set is potential relative to its elements: once the elements exist the set can exist. Take some people: Nadia, Dylan, and Melesha. Since each of them exists, the height potentialist claims that there could have been a set of them: the set {Nadia, Dylan, Melesha} could have existed. Once we have that set, we can repeat the process. Given Nadia, Dylan, and Melesha and the new set, the height potentialist will claim that *they* could have formed a set: the set {Nadia, Dylan, Melesha, {Nadia, Dylan, Melesha}} could have existed. Continuing in this way, we get the possibility of more and more sets. So many, according to the height potentialist, that the sets thus obtained satisfy the axioms of set theory.

Height potentialism is significant because it provides a compelling response to *Russell's paradox*. At its core, Russell's paradox tells us that some conditions do and some conditions do not determine sets. In particular, it tells us that the condition of being a non-self-membered set does not determine a set.[1] We are thus faced with a challenge: provide an account of dividing line between the conditions that do and do not determine sets which avoids problematic cases like the set of non-self-membered sets whilst delivering enough sets for the purposes of mathematics. The account offered by the height potentialism does this extremely well. It says that a condition can determine a set precisely when the things satisfying the condition can co-exist. With the right background assumptions, this can be used to explain why there couldn't have been the problematic sets but why there are enough sets for all of modern mathematics.[2]

Width potentialism is based on the idea that a universe of sets can be used to specify other possible universes of sets. Take, for example, a particular universe of sets $\mathcal{U}$. The width potentialist claims that by applying the method of forcing within $\mathcal{U}$, we can specify other universes of sets: universes in which there are more subsets of the natural numbers than there are in $\mathcal{U}$, for instance. In other words, the claim is that a universe of sets is potential relative to the universes in which it can be specified via forcing: once those universes exist, it *can* exist. According to the width potentialist, there is thus no universe containing absolutely all subsets of the natural numbers and so no universe containing absolutely all sets simpliciter. No universe of sets is privileged on this account: there are many universes, containing different sets, and making different claims true. There is no ultimate background universe of sets, no ultimate V.

[1] Such a set would have to be a member of itself just in case it was not a member of itself!

[2] Height potentialism is most clearly expressed by recent writers like Linnebo [2010], Linnebo and Rayo [2012], Parsons [1977], Studd [2019], and Hellman [1989]. But the view goes back at least to Putnam [1967] and Zermelo [1930]. Arguably, it can be found even in Cantor. See Linnebo [2013] for discussion.

Width potentialism is significant because it provides a compelling response to the *problem of independence*. One of the most important results in modern mathematics is that some of its most fundamental questions are left open by the standard axioms of set theory. The famous example is the continuum hypothesis (CH), which says that every set of real numbers is either countable or has the same cardinality as the reals. CH can neither be proved nor disproved from the currently accepted axioms of set theory. Indeed, despite significant efforts, set-theorists and philosophers have failed to find compelling *new* principles that might prove or disprove it. Width potentialism deals with this problem extremely well. According to the view, the attempt to settle such questions is misplaced. CH is not an unambiguous statement for which we can marshal evidence. Rather, it is true or false only relative to a universe of sets. And in the broad space of universes of sets, we already know how CH behaves: how it is true some universes and false in others. There is no ultimate V in which CH either unambiguously holds or fails to hold.[3]

It is natural to think that height and width potentialism are just aspects of a broader phenomenon of potentialism, that they might both be true.[4] The main result of this paper is that this is mistaken: height and width potentialism are jointly inconsistent. In particular, I will show that the possible sets according to the height potentialist constitute an ultimate background universe of sets, an ultimate V. They contain absolutely all subsets of the natural numbers and in general, whenever they contain an ordinal α, they contain absolutely all subsets of V_α.

Here's the plan. In section 2 I will formulate height and width potentialism and outline their motivations. In section 3 I will give the argument that they are inconsistent, and in section 4 I will consider some responses.

2 Two flavours of potentialism

In this section, I outline the two main flavours of potentialism.

2.1 Height potentialism

Height potentialism is motivated by the paradoxes. To see how, it helps to work in the context of a plural version of Russell's paradox. It has two premises.

First, there's the plural comprehension schema, which says that any condition determines a plurality: for any condition ϕ, there are some things that comprise all and only the ϕs.[5]

[3] Width potentialism finds its clearest formulation in Hamkins et al. [2012]. See also [REFs]. And see Koellner [2006] for a helpful introduction to the problem of independence.

[4] See, for example, Hamkins and Linnebo [forthcoming] and Scambler [forthcoming].

[5] I will frequently use the singular "plurality" and talk of objects being elements or members of pluralities. This is merely for convenience and can always be reformulated using genuinely plural locutions.

Formally:[6,7]

(plural comp) $\exists xx\forall x(x \prec xx \leftrightarrow \phi)$

Second, there is a principle which says that pluralities *collapse* to sets: that any things whatsoever form a set. Formally:

(collapse) $\forall xx\exists x(x \equiv xx)$

where $x \equiv xx$ abbreviates $\forall y(y \in x \leftrightarrow y \prec xx)$. The usual argument for Russell's paradox shows that plural comp and collapse are jointly inconsistent: plural comp delivers a plurality of all and only the non-self-membered sets and collapse then gives us the set formed from that plurality, the set of all and only the non-self-membered sets. Such a set would be a member of itself just in case it was not a member of itself. Contradiction!

So, which assumption should we reject? plural comp is compelling. Pluralities are naturally thought of as nothing over and above the individual things they comprise. On this picture, there is no metaphysical gap between some things taken together and those same things taken individually.[8] Take Nadia, Dylan, and Melesha. The claim is that they are nothing over and above the individuals Nadia, Dylan, and Melesha. So if each of Nadia, Dylan, and Melesha exist, then they exist: there is no way for each of the individuals to exist without the plurality of them existing. Similarly, if each individual ϕ exists, then so too does the plurality of ϕs: there is no way for each of the individual ϕs to exist without the plurality of them existing. The plurality of ϕs is nothing over and above the individual things which happen to be ϕ. In other words, on this conception of pluralities, plural comp holds.[9]

According to the height potentialist, however, there are also compelling arguments in favour of collapse and we are thus faced with a genuine paradox. Their idea is to solve the paradox by claiming that although these arguments *are* compelling, rather than justifying collapse, they justify a similar but importantly weaker claim: namely, the claim that any things *could* have formed a set. Formally:

(collapse$^\Diamond$) $\Box\forall xx\Diamond\exists x(x \equiv xx)$

This modal version of collapse is, unlike collapse itself, perfectly consistent with plural comp. The height potentialist thus proposes to accept both plural comp and collapse$^\Diamond$.

[6]As usual, I will use double variables $xx, yy, zz, ...$ etc to range plurally over whatever the first-order variables range over. The relation symbol $\prec$ is intended to express the relation that an object bears to some things when it is one of them. So $x \prec xx$ is well-formed and read "it_x is one of them_{xx}". I will also assume that the identity symbol can be flanked by plural variables so that "$xx = yy$" is well-formed. Personally, I think good sense can be made of such claims, but as I show in Roberts [ms], they can be eliminated if necessary. Exx will abbreviate $\exists yy(yy = xx)$ and similarly for Ex.

[7]This way of formulating plural comprehension implies that there is an empty plurality. But some have claimed that pluralities must be non-empty: that any things comprise at least one thing. Again, I think good sense can be made of a notion of plurality for which there is an empty plurality—even if that sense differs somewhat from ordinary usage—but this won't affect my arguments. All of the arguments I give can be suitably adjusted to ban empty pluralities. See Linnebo [2013] for discussion.

[8]See Roberts [ms] for discussion.

[9]Though see section 4.3 for discussion.

A crucial question for the height potentialist is how they understand the notion of possibility involved in $\mathsf{collapse}^{\Diamond}$. In what sense *can* any things form a set? Different authors understand the notion in different ways. For some, it is a distinctively mathematical notion.[10] For others, it is interpretational.[11] For some, logical.[12,13]

As important as this question is for the height potentialist, I will mostly ignore it in what follows. All authors agree that the modal logic governing $\Diamond$ should be a version of $\mathsf{S4.2}$ plural modal logic together with suitable background assumptions,[14] and this will suffice for the main result.

Let me now turn to the height potentialist's central argument in favour of $\mathsf{collapse}^{\Diamond}$.[15]

To formulate the argument, it will help to introduce some terminology. Following Studd [2019], say that some things are *collectable* if they could have formed a set. Formally, xx are collectable just in case $\Diamond\exists x(x \equiv xx)$. What $\mathsf{collapse}^{\Diamond}$ says, then, is that any possible plurality is collectable.

To deny $\mathsf{collapse}^{\Diamond}$ is to accept that some possible pluralities are collectable and some are not.[16] But as Studd [2019] points out:

> ...an advocate of [this view] faces an important explanatory challenge: he owes us an explanation of what makes uncollectable pluralities uncollectable (p. 186)

The crucial claim is that the opponent of $\mathsf{collapse}^{\Diamond}$ cannot meet this explanatory challenge in a satisfactory way.

Who is the opponent of $\mathsf{collapse}^{\Diamond}$? My favoured alternative to $\mathsf{collapse}^{\Diamond}$ is the claim that there couldn't have been more sets than there are: that set existence is non-contingent. On this view, for some things to possibly form a set is for them to actually form a set and $\mathsf{collapse}^{\Diamond}$ becomes equivalent to $\mathsf{collapse}$. So, since the non-self-membered sets don't actually form a set, they couldn't have formed a set and $\mathsf{collapse}^{\Diamond}$ is false. In general, the view implies that when we restrict our attention to claims solely about sets and pluralities, the modality becomes redundant.[17] Call this view *actualism*. Formally:

(actualism) $$\exists xx\Box\forall x(x \prec xx)$$

[10] Linnebo [2013], Parsons [1983b], Hellman [1989], Reinhardt [1980].

[11] Linnebo [2018], Studd [2019], Uzquiano [2015].

[12] Hellman [1989], Berry [2018].

[13] Most reject a metaphysical interpretation. Sets, the thought goes, are abstract objects and therefore exist of metaphysical necessity if at all. In that case, $\mathsf{collapse}^{\Diamond}$ would imply $\mathsf{collapse}$, resulting in inconsistency. But it is not at all obvious that we have to accept the assumption that mathematical objects do exist of metaphysical necessity (REF Rosen, PETTIGREW [2012]). To me, at least, using metaphysical possibility to make sense of the modal operator is a live option.

[14] See sections 3 and 4 for details.

[15] See Linnebo [2010] and Studd [2019].

[16] Assuming there could have been some sets.

[17] More precisely, we can show (given suitable background assumptions) that given $\mathsf{actualism}$:

$$\phi \leftrightarrow \Box\phi \leftrightarrow \Diamond\phi$$

See Roberts [2016] chapter 1 for details.

For our purposes, we can take the crucial height potentialist claim to be that the actualist does not have a satisfactory response to the explanatory challenge. Why? The reason is that neither of the two standard ways for the actualist to meet the challenge—using the *limitation of size* or *iterative* conceptions of set—provides a satisfactory response.

According to the limitation of size conception of sets, some things form a set precisely when they are fewer than the ordinals. The ordinals thus provide a threshold cardinality below which pluralities form sets and above which, they don't. On this view, the most natural response to the explantory challenge is to claim that what makes the uncollectable pluralities uncollectable is that they are "too large": that they are not fewer than the ordinals.

According to the iterative conception of sets, the sets occur in a well-ordered series of stages. At the very first stage, we have no sets whatsoever. Then, at the second stage, we have all the sets of things at the first stage: that is, since there is nothing at the first stage, we have the empty set! At the third stage, we have all the sets of things at the second stage: that is, since the empty set is the only thing at the second stage, we have precisely the set containing the empty set and the empty set itself. At the fourth stage, we have all the sets of *those* things. And so on indefinitely. In general, at any stage we have sets of any things which all occur together at some previous stage. On this view, some things form a set precisely when they all occur together at some stage and so the most natural response to the challenge is to claim that what makes an uncollectable plurality uncollectable is that there is no stage at which its elements all exist together.

The charge is that each of these responses fails in important cases. For example, we know that the ordinals do not form a set. According to the limitation of size response:

> the explanation is that [the ordinals] are too many to form a set, where being too many is defined as being as many as [the ordinals]. Thus, the proposed explanation moves in a tiny circle. [Linnebo, 2010, p.154]

Similar claims are made for the iterative conception of sets.[18]

[18]The same claim for the iterative conception is unpersuasive. According to that conception, the reason why the ordinals do not form a set is that they do not all co-exist at some stage. In contrast to the limitation of size conception, this proposed explanation doesn't seem at all circular. Linnebo [2010] argues that the unexplanatoriness of the limitation of size conception is inherited by the iterative conception. As he shows, given plausible assumptions that the proponent of the iterative conception should accept, the actualist is committed to the claim that some things form a set just in case they are fewer than the ordinals. The problem with this is that the proponent of the iterative conception need not think this claim carries and *explanatory* weight. I may be committed to the claim that Zara is fond of all and only the pluralities that form sets. It doesn't follow that I'm committed to the claim Zara's fondness for a plurality explains why it does or does not form a set. Explanation is a highly intensional notion and materially equivalent statements will not always explain the same things.

Studd [2019] focuses on a different example and a different formulation of the iterative conception. According to his preferred formulation, the sets are arranged in the V_α hierarchy. The proposed explanation of why some things form a set is then that the rank of each of them is bounded by some ordinal. Now, the finite ordinals form a set. This is just what the axiom of infinity tells us. So the explanation would be that this is because there is some ordinal greater than the ranks of each of the ordinals. Since the rank of an ordinal is that very ordinal, that means there is some ordinal greater than each finite ordinal. But that is precisely what it mean for them to form a set! The problem with this is that it relies on a very specific way of formulating the iterative conception. On my preferred way of formulating it, we take the notion of stage to be primitive but we take stages to be a certain kind of set. (Since we might not think that this explains why the things in a stage form a set, we could also take stages to be a certain kind of plurality.) Then the explanation is that the finite ordinals form a set because they all co-exist at some stage. Again, this doesn't seem at all circular. Of course, The actualist faces an explanatory challenge that they fail to meet. The height potentialist faces no such challenge. Other things being equal, height potentialism should be preferred.

That's the central argument for $\mathsf{collapse}^\Diamond$.[19] Since we have independent reason to adopt $\mathsf{plural\ comp}$, we have reason to adopt both $\mathsf{collapse}^\Diamond$ and $\mathsf{plural\ comp}$. Height potentialists typically go further, though. They typically think that the sets we get via $\mathsf{collapse}^\Diamond$ satisfy the axioms of standard set theory.[20] Indeed, they see it as one of jobs of height potentialism to *motivate* the axioms of standard set theory by showing how they hold in the potential sets.[21] For a number of those axioms, this is almost immediate: the axioms of pairing, union, and separation all follow in the background modal logic alone.[22] The axioms of foundation, infinity, powerset, and replacement, however, require further assumptions. Since the height potentialist's central argument for $\mathsf{collapse}^\Diamond$ is broadly abductive, it may seem necessary that they adopt some such assumptions. The axioms of pairing, union, separation, foundation, and powerset, for example, are all easy consequences of the actualist iterative conception.[23] So without those further assumptions, the explanatory benefit of $\mathsf{collapse}^\Diamond$ might be outweighed by the actualist iterative conception's ability to imply and explain the axioms of foundation and powerset.[24]

In any case, I will now show that the central argument for $\mathsf{collapse}^\Diamond$ generalises to an argument for what will come to be an important fragment of the axioms of ZFC.

Say that some things are *collected* just in case they form a set. Formally, xx are collected precisely when $\exists x(x \equiv xx)$. What $\mathsf{collapse}$ tells us, therefore, is that every plurality is collected. Since both the actualist and the height potentialist accept $\mathsf{plural\ comp}$, they both deny $\mathsf{collapse}$. For both theorists, some pluralities are collected and some aren't.[25] Indeed, this holds of necessity: necessarily some pluralities are collected and some aren't.[26] We are thus faced with *another* explanatory challenge: we are owed an explanation of what makes the uncollected pluralities uncollected in a given world.

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with the right assumptions, this formulation of the iterative conception will *imply* Studd's formulation. But as above, the proponent of the iterative conception need not put any explanatory weight on that formulation.

[19] I won't assess the force of the central argument here, though see Roberts [2016] for a negative assessment.

[20] They make this precise using the notion of *modalisation*. The modalisation of a formula in the language of set theory is just the result of replacing its quantifiers with the corresponding *modalised quantifiers*: that is, the result of replacing occurrences of $\exists x$ with $\Diamond\exists x$ and $\forall x$ with $\Box\forall x$. This has the effect of forcing its quantifiers to range over all possible sets, as required. Let $\phi^\Diamond$ be the modalisation of ϕ and let $\mathsf{ZFC}^\Diamond$ be the modalisations of the axioms of ZFC. They then prove the following "mirroring" theorem which shows that modalisations behave logically just like their unmodalised forms. This shows that we can talk about what's true in the potential sets without assuming they constitute a plurality.

Theorem 1 ϕ is provable in first-order logic from Γ precisely when $\phi^\Diamond$ is provable in $\mathsf{S4.2}$ modal logic from $\Gamma^\Diamond$ (plus background assumptions).

See, for example, Studd [2019] and Linnebo [2010], Linnebo [2013], etc.

[21] See, for example, [Studd, 2019, p.??].

[22] [details]

[23] As is the claim that every set is in some V_α.

[24] Studd [2019], for example, obtains the powerset axiom by adopting a stronger version of collapse (see below). Linnebo [2013] simply adds the powerset axiom as a further assumption. Studd [2019] secures foundation via a modal principle, whereas Linnebo [2013] puts it in by hand. Both employ a *reflection principle* to obtain infinity and replacement. Such a principle is also available to the actualist, and is just as well motivated.

[25] Assuming that some set exists.

[26] Assuming it is necessary that some set exists.

As I mentioned above, the modality expressed by $\Diamond$ is effectively redundant for the actualist: collapse$^\Diamond$ is equivalent to collapse and to be collectable is to be collected. So the two explanatory challenges are equivalent for them. They face, in other words, one explanatory challenge that can be formulated either in terms of being collectable or in terms of being collected. Their response to both, we can assume, is derived from the limitation of size or iterative conceptions, both of which fail to be explanatory in certain crucial cases according to the height potentialist. The modality is certainly not redundant for the height potentialist: collapse$^\Diamond$ is inequivalent to collapse—the first true, the second false. The two explanatory challenges are thus inequivalent for them also. And although they sidestep the challenge to explain what makes the uncolletable pluralities uncollectable—for they think there could not have been any such pluralities—they face the challenge to explain what makes the uncollected pluralities uncollected head on.

It can be shown that dividing line between the collected and uncollected pluralities varies wildly between models of plural comp and collapse$^\Diamond$ and indeed between worlds within a single model.[27] So without supplementation, height potentialism tells us very little about which pluralities are uncollected in a given world, let alone why they are uncollected. Clearly, an appeal at this point to either the limitation of size or iterative conceptions by the height potentialist would leave their earlier argument undermined. For the proposed explanations would be precisely the same as those offered by the actualist. Each would be equally unexplanatory. The actualist would effectively face one challenge—since both are equivalent—and give a somewhat unexplanatory response and the height potentialist would effectively face one challenge—since one does and one doesn't apply to them—and give an equally unexplanatory answer. A stalemate.

Luckily, the potentialist has an alternative response to the second explanatory challenge, but it requires further modal resources. It is based on the idea that the elements of a set are prior to the set: that the elements of a set must have been available *before* the set. To make this idea precise, we need another modal operator: one that expresses the crucial notion of "before", a dual to $\Diamond$ that "looks back" instead of "forward". Formally, we can add to our language a pair of operators $\Diamond^<$ and $\Box^<$ meaning roughly that it will and must be the case respectively and a pair of operators $\Diamond^>$ and $\Box^>$ meaning that it was and always was the case respectively. Let $\boxdot$ be an operator which says that it always was, is, and always will be the case. Formally, $\boxdot\phi$ just in case $\Box^<\phi \wedge \phi \wedge \Box^>\phi$. Following Studd [2019], the priority idea can then be expressed as follows.[28]

(priority) $$\boxdot\forall x \Diamond^> \exists xx(x \equiv xx)^{29}$$

[27]For example, there are models in which the worlds are inaccessible ranks but also models in which the worlds are just the transitive sets. See Hamkins and Linnebo [forthcoming] for further examples.

[28]In addition to allowing for a response to the explanatory challenge, Studd [2019] argues that the bi-modal setting is preferable to the uni-modal setting for other reasons. For example, it allows us to derive the axiom of foundation from a compelling bi-modal principle.

[29]There are subtle issues which arise in the bi-modal setting that don't arise in the uni-modal setting. For example, in the uni-modal setting we are able to adopt an underlying classical logic (Linnebo [2013]). It thus follows in that setting that sets exist necessarily. So we are not confronted with the question what members a set has in worlds where it doesn't exist. In the bi-modal setting, on the other hand, we can "look back" to worlds where given sets do not exist. Suppose, for example, that in worlds where x doesn't exist, it has no members. Then priority will be trivially true for x because there will be a prior world at which x doesn't exist and thus at which it is co-extensive with the empty plurality. The simplest way to deal with this is by

How does this help with the challenge? Well, it places an upper bound on the pluralities that are collected at a given world. It says that only those pluralities whose elements all co-exist at some prior world form sets at the given world. But it does not give us a lower bound. $\mathsf{collapse}^{\Diamond}$ ensures that any things will form a set at some later world, but it does not tell us when: we may have to pass through many worlds before we get to it. Fortunately, a natural strengthening of $\mathsf{collapse}^{\Diamond}$ does give us a lower bound. It says that once some things exist, the set of them *must* exist. In other words, any plurality whose elements all co-exist at some prior world form a set at a given world. Formally:

(plenitude) $$\Box\forall xx\Box^{<}\exists x(x \equiv xx)$$

Together, then, priority and plenitude tell us that the pluralities which are collected at a given world are precisely those whose elements all co-exist at some prior world. Formally:

$$\Box\forall xx(\exists x(x \equiv xx) \leftrightarrow \Diamond^{>}Exx)$$

Since plenitude implies $\mathsf{collapse}^{\Diamond}$, we get a response to both challenges. The purported explanation for why the ordinals in a world don't form a set, for example, is that they hadn't all co-existed at some prior world. This does not seem to suffer from the same shortcomings as the purported explanation given by the actualist.[30]

The central argument for $\mathsf{collapse}^{\Diamond}$ thus seems to generalise to an argument for priority and plenitude and the modalities needed to formulate them. Those principles, in turn, imply the modalisiations of a large fragment of ZFC. As Studd [2019] shows, given the axiom of infinity, they imply the modalisations of the axioms of Zermelo set theory (Z) together with the claim that every set is in some V_α rank ($\mathsf{Z} + \forall x\exists\alpha(x \in V_\alpha)$). It is this last claim that will be crucial for our purposes. It tells us that the potential sets have a uniform structure, the same structure as any model of ZFC: namely the structure of the V_α ranks. Standard extensions of this theory differ only in *how far* they take those ranks to extend. For example, ZFC adds the axiom of replacement which is equivalent to the claim that the ranks extend further than the size of any particular set. For this reason, I claim that the truth of the axioms of $\mathsf{Z} + \forall x\exists\alpha(x \in V_\alpha)$ is a guarantee that we are dealing with a universe of sets rather than an arbitrary collection of sets.[31] If all of this is right, it follows that the height potential

replacing occurrences of $x \in y$ throughout the theory with $\Diamond(x \in y)$ as Studd [2019] does: although a set can change its elements from world to world, like x, it cannot change its possible elements. For simplicity, though, I will ignore this complication.

[30]Nonetheless, a related question still remains: namely, what pluralities are collectable simpliciter—that is, in the actual world? Alternatively, why are *these* the actually existing sets, rather than the sets in some other world? The current proposal says that it is precisely those sets whose elements all co-exist at some prior world. But that is of little help unless we know which worlds are prior to the actual world, which in effect is the very question we are trying to answer. It is consistent with priority and plenitude that any world in a given model is the actual world. There is a *non-arbitrary* answer to the question: namely, that there are actually no sets, that the actual world is the very first world. But this would not *explain* why the world is that way, why it is devoid of sets.

I take this explanatory challenge to be extremely important in the debate between the height potentialist and the actualist since it is unclear how the height potentialist can meet it. The actualist, on the other hand, does not face the challenge. According to them, there are no other ways the actual world could be with respect to what sets exist. Set existence is non-contingent. The balance of explanatory virtue may not tip so heavily in favour of the height potentialist after all. See Roberts (ms) for further discussion, and Menzel ???? for a similar point.

[31]We may also require transitivity.

sets constitute a universe of sets. Later, I will argue that they constitute an ultimate universe of sets, an ultimate V.

2.2 Width potentialism

The core claim of width potentialism is that, by applying the method of forcing within a given universe of sets, we can specify other possible universes of sets. Possible universes, for example, with more subsets of the natural numbers than the given universe. Ostensibly, then, width potentialism concerns universes of sets and possibility. For now, we can ignore the modal aspect and take its core claim to be merely about universes of sets, for which I will use variables $\mathcal{U}, \mathcal{U}', \mathcal{U}''$. We will return to the question how we should understand the notion of a universe of sets, but, for now, we can just assume that they are collections of sets which satisfy the axioms of ZFC.

To get clearer on the view, it will help to look at the basics of the method of forcing. Let $\mathcal{U}$ be a particular universe of sets. Within $\mathcal{U}$ there will be many *partial orders*: that is, sets which combine a domain of objects together with a non-reflexive, transitive relation $<$ over them. Let $\mathbb{P}$ be any such partial order. I will use variables $p, q, r, ...$ etc for the elements of $\mathbb{P}$. To my mind, the most natural way to think of these elements is as *possibilities*.[32] Possibilities are like *parts* of possible worlds. Whereas possible worlds are complete specifications of ways the world could be, a possibility is a partial specification of a way the world could be. For example, p might be the possibility that the cat is on the mat. It is part of any world in which the cat is on the mat, but it is consistent both with the possibility that the cat is black and with the possibility that it is not black. On this way of thinking, for one possibility, q, to be less than another, p, in the partial order, $q < p$, is for q to extend p with new information. For example, p might be the possibility that the cat is on the mat and q the possibility that the cat is on the mat *and* black. Thus, as we move "down" the partial order, we get more information and the possibilities become more specific. When one possibility q extends another p, I will also say that p is a *part* of q.

Once we think of the elements of our partial order as possibilities, we can use them to construct *intensional* models of the language of set theory. First, we decide what the possible sets of the model will be. Then, for our basic relations—that is, set membership and identity—we decide which of the possible sets relate to which relative to which possibilities. For example, suppose we have two possible sets x and y. Then we can define an intensional model by stipulating that p is a possibility on which x is an element of y and that p is the only such possibility. This would be a model in which x is an element of y precisely when the possibility p is the case. In other words, we can generate intensional models by supplying a *domain* of objects D and two *interpretation functions* $I_\in$ and $I_=$ which map ordered pairs of elements of the domain to sets of possibilities. Suppose, for example, that our interpretation function $I_\in$ assigns the set of all possibilities to a pair $\langle x, y\rangle$: that is, $I_\in(\langle x, y\rangle) = \mathbb{P}$. Then according to that assignment, the claim that x is a member of y is *necessary.* No matter how things had been, x would have been an element of y; x is an element of y according to all possibilities. Conversely, suppose our interpretation function assigns the empty set of possibilities to $\langle x, y\rangle$: that is, $I_\in(\langle x, y\rangle) = \emptyset$. Then according to that assignment, the claim that x is a member of y is *impossible.* No matter how things had been, x wouldn't have been an element of y; x is an element of y according to no possibilities.[33]

[32] [Possibility semantics REFs.] [Have others proposed thinking about forcing like this?]

[33] It may be instructive to compare these models with classical, non-intensional, models. There we also have

We can think of sets of possibilities as *propositions*: the proposition that one of those possibilities obtains. So, an interpretation function can be seen as assigning propositions to order pairs over the domain. There is then a natural way to compositionally extend this to all claims in the language of set theory. Let $[\![\phi]\!]$ be the proposition assigned to ϕ. Compositionally, we let $[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$, $[\![\exists x \phi]\!] = \bigcup_{x \in D} [\![\phi]\!]$, and $[\![\neg \phi]\!] = \{p :$ p is incompossible with every possibility in $[\![\phi]\!]\}$ (where p is *compossible* with q when there is some possibility that has them both as parts—that is, some r for which $r < p$ and $r < q$—otherwise, they are *incompossible*). In other words, the possibilities for a disjunction are the possibilities for each disjunct; the possibilities for an existential claim are the possibilities for its instances; and the possibilities for the negation of ϕ are the possibilities that are incompossible with the possibilities for ϕ.

Propositions come equipped with a natural notion of implication. In particular, we can say that a proposition $X \subseteq \mathbb{P}$ implies another proposition $Y \subseteq \mathbb{P}$—in symbols, $X \vDash Y$—when it is impossible for X to be true but Y false. That is, there is no possibility that has an X possibility *and* a $\neg Y$ possibility as parts.[34] It is then straightforward to show that the laws of classical first-order logic without identity are respected: if an argument from $\phi_0, ..., \phi_n$ to ψ is valid in first-order logic without identity, then $[\![\phi_0 \wedge ... \wedge \phi_n]\!] \vDash [\![\psi]\!]$.[35] Moreover, by picking a suitable interpretation function $I_=$, we can also ensure that the identity axioms are necessary so that all arguments of first-order classical logic are preserved. In what follows, I will assume that we have done so.

It turns out that there is a canonical intensional model definable within any universe of sets $\mathcal{U}$. It is characterised by three core properties: maximality, well-foundedness, and extensionality. maximality says that every specification of a possible set defines a possible set. In other words, given any set of possible sets X and any way f of assigning propositions to those sets, there will be a possible set s for which membership is completely determined by X and f. More precisely, there is an s for which $[\![x \in s]\!] = [\![(x = z_0 \wedge f(z_0)) \vee (x = z_1 \wedge f(z_1)) \vee ... \vee (x = z_i \wedge f(z_i))...]\!]$ for $z_i \in X$. To be an element of s, in other words, just is to be equal to some z in X and for its associated proposition to be true. When membership in s is definable from X in this way, we say that X a *core* of s.[36]

For classical models, well-foundedness is the claim that there are no infinitely descending chains of membership: that there is no sequence $x_0 \ni x_1 \ni x_2, ..., \ni x_n, ...$ For intensional models, it comes to the claim that there are no infinitely descending chains of membership *across possibilities*. But it turns out that we have to be quite cautious in how we make this

a domain and interpretation function, but the interpretation function only maps pairs to the truth values: true and false. By looking at the clauses for truth in an intensional model below, it is easy to see that classical models are a special case of an intensional model: they can be identified with intensional models where $\mathbb{P}$ only contains *one* possibility. To be possible is then just like being true simpliciter and to be impossible is just like being false simpliciter. One important difference between intensional and classical Kripke models is that we allow I to provide a non-trivial interpretation of the identity relation. Unlike classical Kripke models, identity can be contingent: the set of possibilities assigned to a pair may neither be the set of all possibilities nor the set of none. [Compare with Kripke models more generally]

[34]Formally: $\neg \exists p \exists q, r(p < q, r \wedge q \in X \wedge r \in [\![\neg Y]\!])$. In other words, if any possibility containing an X possibility can be extended to contain a Y possibility. Formally: $\forall p \in X \forall q(q < p \rightarrow \exists r < q \exists s \in Y(r < s))$.

[35]Indeed, in general, the implication relation gives rise to a complete Boolean algebra (modulo implicational equivalence). So, I will sometimes talk about infinite conjunctions and disjunctions.

[36]The notion of a core can be characterised without reference to f because we can show that X is a core for s (relative to any f) precisely when $[\![x \in s]\!] = [\![(x = z_0 \wedge z_0 \in s) \vee (x = z_1 \wedge z_1 \in s) \vee ... \vee (x = z_i \wedge z_i \in s)...]\!]$ for $z_i \in X$. A core for s is thus just a set membership of whose elements in s determines membership of any possible set in s.

precise. Consider the following example. Given maximality, there will be a possible set, $\emptyset$, that never contains anything: it is necessarily empty. Let p and q be incompossible possibilities. Then, again by maximality, there will be two possible sets s_1 and s_2 such that s_1 contains $\emptyset$ if p is the case and otherwise contains nothing and s_2 contains $\emptyset$ if q is the case and otherwise contains nothing. So, according to p, $\emptyset$ is an element of s_1 and $s_2 = \emptyset$ and according to q, $\emptyset$ is an element of s_2 and $s_1 = \emptyset$. Thus, according to some possibility—namely, p—s_1 is an element of s_2, and according to some other possibility—namely, q—s_2 is an element of s_1. Which seems to violate well-foundedness.

There are two very different ways of describing s_1 and s_2. On one description, the only element they can have is $\emptyset$ and that can't in turn have any elements. Under this description, they are both well-founded. On the other description, s_1 is the set which contains s_2 if p is the case and otherwise contains nothing and s_2 contains s_1 if q is the case and otherwise contains nothing. Under this description, they aren't well-founded. In other words, there are two ways of assigning cores to s_1 and s_2: one relative to which they are well-founded and one relative to which they aren't. The well-foundedness property says that there is some uniform way of describing the possible sets relative to which they are well-founded. More precisely, the claim is that there is a function F from possible sets to cores such that there is no descending chain $x_0, ..., x_n, ...$ for which $F(x_n) \ni x_{n+1}$.

extensionality is much simpler. It just requires that the axiom of extensionality—which says that sets with the same elements are identical—is necessarily true. We can then prove that any two intensional models that satisfy maximality, well-foundedness, and extensionality are isomorphic modulo identity.[37,38]

Now the central result driving width potentialism is that given any partial order $\mathbb{P}$ in any universe $\mathcal{U}$, there is a (definable) intensional model satisfying maximality, well-foudnedness, and extensionality.[39] Moreover, we can show that in any such model the axioms of ZFC are necessary. That is, $[\![\phi]\!]$ is the necessarily true proposition whenever ϕ is an axiom of ZFC. Furthermore, depending on what partial order we consider, various other claims can be made necessary. For example, if our set of possibilities is the set of finite functions from ω_2 to 2, then it will be necessary that the continuum hypothesis is false.[40] This is how forcing is standardly used to show that the continuum hypothesis is not disprovable in ZFC.

By thinking of the elements of a partial order as possibilities, the associated canonical intensional model can be thought of as specifying a range of ways the universe of sets could be. The crucial claim behind width potentialism is that this specification succeeds: some universe $\mathcal{U}'$ is one of those ways. What does it mean for a universe to be one of the ways so specified? Since possibilities are parts of possible worlds, possible worlds can be identified with certain maximally specific sets of possibilities. More precisely, possible worlds can be identified with sets of possibilities that are *coherent*—any two possibilities that it contains should be compossible—and it should be *complete*—for any proposition X in $\mathcal{U}$, it either contains a possibility in X or in $[\![\neg X]\!]$.[41] Let $W \subseteq \mathbb{P}$ be a possible world. We say that a

[37]In other words, if we modify the models so that necessarily identical sets are identical, then we get isomorphic models.

[38][Explain how this is just the intensional version of the quasi-categoricity theorems]

[39]The usual way to do this is by transfinite recursion, defining so-called names—which are just the possible sets of the model—and then suitable interpretation functions $I_\in$ and $I_=$ over them. See any textbook on forcing for details. REF.

[40]REF

[41]In the forcing literature, possible worlds in this sense are called *generic filters*.

proposition X is true in W just in case W contains a possibility p which implies X—that is, $\{p\} \vDash X$.[42,43] Then we can define a classical model $\mathcal{M}_W$ as follows. Its domain is just the domain of our canonical model and we stipulate that $\mathcal{M}_W \vDash x \in y$ iff $[\![x \in y]\!]$ is true in W and $\mathcal{M}_W \vDash x = y$ iff $[\![x = y]\!]$ is true in W. It is then straightforward to show that in general:

$$\mathcal{M}_W \vDash \phi \leftrightarrow [\![\phi]\!] \text{ is true in } W$$

Since the axioms of ZFC are necessary and since the negation of the necessary proposition is empty, each of those axioms will be true in $\mathcal{M}_W$. For the same reason, it will satisfy the identity axioms. Finally, let $\mathcal{M}_W^{=}$ be $\mathcal{M}_W$ modulo identity. It is then a standard logical result that $\mathcal{M}_W^{=}$ and $\mathcal{M}_W$ make the very same sentences true.

So, the crucial width potentialist claim can be formulated precisely as the claim that for any universe $\mathcal{U}$ and any partial order $\mathbb{P}$ in $\mathcal{U}$, there is some $\mathcal{U}'$ which is isomorphic to $\mathcal{M}_W^{=}$ for some possible world W.[44]

By a judicious choice of partial order, we can use the claim to obtain universes with more subsets of the natural numbers than $\mathcal{U}$. More generally, we can use it to obtain universes with more subsets of any set in $\mathcal{U}$.[45] Indeed, we can use the central claim to show that for any set x in $\mathcal{U}$, there is a universe in which x is *countable*. Moreover, we can show that whenever $\mathcal{U}'$ is obtained in this was from $\mathcal{U}$, it has precisely the same ordinals as $\mathcal{U}$. Since every universe has the V_α structure, that means the only way to add sets is by adding subsets of some V_α. Hence *width* potentialism rather than *height* potentialism.

The arguments for width potentialism are less than clear. For example, Hamkins et al. [2012] presents what many take to be the central case. His main argument is that if we deny width potentialism, then:

> [we] must explain or explain away as imaginary all of the alternative universes that set-theorists seem to have constructed. This seems a difficult task, for we have a robust experience in those worlds, and they appear fully set-theoretic to us.

It's hard to know what to make of this. Of course, set-theorists don't literally construct sets or universes of sets. And is it common ground that there are many set-theoretic models of set theory, some of which are forcing extensions of others.[46] The thought must thus be that

[42] It turns out this is equivalent to W containing a possibility in X: that is, X is true in W just in case $X \cap W \neq \emptyset$.

[43] For many $\mathbb{P}$, W provably cannot exist in $\mathcal{U}$. *Proof:...*

[44] We need not worry about the notion of isomorphism here, since we can equally formulate the crucial claim without it. In particular, we can equally say that $\mathcal{U} \subseteq \mathcal{U}'$, $W \in U'$, and for all $x \in \mathcal{U}'$ there is $y \in \mathcal{U}$ in the domain of the canonical intensional model such that $x = y^W$ where y^W is defined by recursion in $\mathcal{U}'$ in the following way... See any forcing textbook for details.

[45] Indeed, for any set in a universe, we can find another universe in which that set is countable. It is interesting to note that the width potentialist claim is equivalent, given plausible assumptions, to the much simpler claim that any set in any universe is countable in some other universe. One direction is the just the observation above. The other direction is obtained by noting that given a universe $\mathcal{U}'$ in which the subsets of $\mathbb{P}$ in $\mathcal{U}$ is countable, we can explicitly define a possible world W using ZFC. We just enumerate the propositions and ... Then we can define in $\mathcal{U}'$ the collection $\{x : \exists y \in \mathcal{U}(x = y^W)\}$. If that is a universe, then we are done. So assuming that inner models of universes are universes, we get the equivalence.

[46] In particular, the width potentialist claim is true when we think of universes as countable transitive models of ZFC.

set-theorists believe, via the method of forcing, that there are alternative universes of sets. The problem is that it is unclear that set-theorists on the whole do believe this and it is unclear what evidence it would constitute if they did.[47]

So rather than focusing on the evidence for width potentialism, I want focus on one of its central applications: namely, to the problem of independence. The problem of independence arises from the well-known fact that there are fundamental questions in the language of set theory which are left open by the axioms of ZFC. The continuum hypothesis (CH) is perhaps the most famous example. It says that there are no sizes between the size of the natural numbers and the size of the real numbers and it is neither provable nor disprovable from the axioms of ZFC. If there is an ultimate background universe of sets in which these various claims are either true or false, then we are faced with a challenge: to formulate new axioms which settle them one way or the other and for which we have good evidence. Despite significant efforts, this challenge has gone unanswered. Our best candidates for new axioms—namely, large cardinal hypotheses—all fail to prove or disprove CH.[48]

Width potentialism rejects the assumption on which the challenge is based: there is no ultimate background universe of sets, no ultimate V. CH is only true or false relative to some one of the myriad possible universes. And in the space of those possible universes, we already know how it behaves: how it is true in some universes and false in others, and how forcing can be used to switch between them.[49] But this response only works if it is sufficiently general. For example, suppose we could take the union $\bigcup\mathcal{U}$ of all the width potentialist universes and suppose furthermore that its sets satisfied the axioms of ZFC. Technically, $\bigcup\mathcal{U}$ wouldn't be a width potentialist universe since it contains all the width potential subsets of the natural numbers. But $\bigcup\mathcal{U}$ looks like just as good a candidate for a universe of sets. Indeed, since it contains every set in every width potentialist universe of sets, it looks like it might be a good candidate for an ultimate background universe of sets. The question whether CH holds in $\bigcup\mathcal{U}$ seems as pressing and as hard as the question whether CH holds simpliciter. To sustain the response to the problem of independence, width potentialism needs to have broad applicability. If a collection of sets appears for all intents and purposes like a universe of sets, then it should be counted as such by the width potentialist.

3 Height and width potentialism are inconsistent

In this section, I will argue that height and width potentialism are jointly inconsistent. I will start with a core result. It says that given plausible background assumptions, if the possible$_\Diamond$ sets are closed under collapse$^\Diamond$ and plural comp, then it is impossible, *in any sense*, to add subsets to them. It is an immediate consequence that the possible$_\Diamond$ sets do not constitute a

[47]It's unclear, in particular, that in believing this they aren't moving beyond their area of expertise. And if they are, there's very little reason to take that as evidence.

[48]See, e.g., Koellner [2006].

[49]See Hamkins et al. [2012] for discussion. Of course, the width potentialist faces a version of the challenge, since there are all sorts of questions which will remain independent of an axiomatisation of their view. For example, whether there are universes with various large cardinals (say, Reinhardt cardinals without the axiom of choice). Or suppose we can make sense of functions across universes—using second-order logic, for example. Then we can formulate the question: are there as many sets of natural numbers as there are countable ordinals? (Since every ordinal can be forced to be countable, this comes to the question whether there is a one-one function between the ordinals and the sets of natural numbers). It's not at all clear that the width potentialist is in a better position to answer this than someone who thinks there is a single universe. The assumption is that this challenge is somehow less tricky or less important.

universe of sets according to the width potentialist. But as we've seen, that is precisely what the height potentialist maintains. More strikingly, I will use the core result to argue that given height potentialism, the possible$_\Diamond$ constitute an ultimate universe of sets, an ultimate V, at least up to their height.

In essence, the core result is straightforward and the basic ideas are well-known. Indeed, the key idea is easily explained. Let $\mathcal{U}^\Diamond$ be the collection of all possible$_\Diamond$ sets and let x be one of its sets. Now suppose there is some subset y of x which is not in $\mathcal{U}^\Diamond$. Let yy be the plurality of elements in y. Since x is in $\mathcal{U}^\Diamond$ and y is a subset of x, the yy are a subplurality of x and thus exist in $\mathcal{U}^\Diamond$. But by $\mathsf{collapse}^\Diamond$, the yy form a set z in $\mathcal{U}^\Diamond$. So, z is a set in $\mathcal{U}^\Diamond$ with precisely the same elements as y, namely the yy. But sets are *extensional*: sets with the same elements are identical. So, $z = y$ and y is in $\mathcal{U}^\Diamond$ after all. Contradiction![50] The core result just makes this simple proof idea fully precise.

3.1 The core result

Since height and width potentialism concern the possible existence of sets, modal logic is the most natural setting in which to state and prove the core result.[51]

We will work in the language of plural set theory supplemented with three modal operators: $\Diamond$, $\blacklozenge$, and @. We can think of $\Diamond$ as the height potentialist's modality and $\blacklozenge$ as the width potentialist's modality. But formally we just assume that both modal operators are governed by the minimal modal logic K. The actuality operator is intended to be used in such a way that it *rigidly* refers to the actual circumstances. This will allow us to avoid unhelpful interactions between the other two modal operators. I will assume it is governed by the modal logic K together with the usual axioms for rigid actuality, as follows.

$$@@\phi \leftrightarrow @\phi$$

$$\Diamond @\phi \leftrightarrow @\phi$$

$$\blacklozenge @\phi \leftrightarrow @\phi$$

$$@\neg\phi \leftrightarrow \neg @\phi$$

Finally, we're only going to be interested in what's actually possible in one of these senses, so we will add the following stipulative assumption.[52]

[50]REF Martin, Zermelo.

[51]As we will see in the next section, however, it also has applications in non-modal settings.

[52]We also adopt a positive free logic for the first-order and plural quantifiers. See, for example, Roberts [2019] for details. A free logic is necessary in this setting because of the actuality operator. In classical modal logic we can derive the claim that necessarily everything is actually something. Classical logic has Ex as a theorem, from which we can obtain $@Ex$ by necessitation, then $\forall x @Ex$ by universal generalisation, and finally $\Box\forall x @Ex$ by one more application of necessitation. In other words, classical modal logic entails a version of $\mathsf{actualism}$. The height potentialist must thus opt for such a free logic. In the absence of an actuality operator, classical modal logic can be used as a framework for height potentialism. See, for example, Linnebo [2013]. But as these observations show, it is not a *robust* framework for the view. Similarly, as Roberts [2016] shows, height potentialism is inconsistent over classical logic with the $UG\Box\forall$ rule of inference. This highly desirable principle effectively allows us to reason with existential witnesses within the scope of modal operators. It is required, for example, to show that when $\phi \wedge \Diamond(Ex \wedge \psi) \to \chi$ is a theorem, then so is $\phi \wedge \Diamond\exists x\psi \to \chi$ (where x is not free in ϕ or χ).

$$\phi \to @\Diamond\phi \vee @\blacklozenge\phi$$

Let $\Diamond^*$ be the disjunction of these modalities. That is:

$$\Diamond^*\phi =_{df} @\Diamond\phi \vee @\blacklozenge\phi$$

It is straightforward to verify that given the above assumptions, $\Diamond^*$ will obey the axioms of S5.

In addition to the logic, we need some assumptions about sets and pluralities. I will return to their motivations in section 4. For now, I will just state them. The first three concern the modal behaviour of sets and pluralities. They express the idea that sets and pluralities are completely determined by their elements. First, we have an extensionality principle which says that sets with the same elements are identical. Formally:

(set-ext) $$\Box^*\forall x\Box^*\forall y(\Box^*\forall z(\Diamond^*(z \in x) \leftrightarrow \Diamond^*(z \in y)) \to x = y)$$

Second, we have the assumption that a set cannot exist without its members and that it cannot change its membership across worlds where it exists. Formally:

(set-rigidity) $$x \in y \to \Box^*(Ey \to Ex \wedge x \in y)$$

For pluralities, we need the analogue of this claim: whenever some things comprise an object, they cannot exist without that object and without comprising that object. Formally:

(plural-rigidity) $$x \prec xx \to \Box^*(Exx \to Ex \wedge x \prec xx)$$

Finally, we need an instance of plural comprehension. It says that given a set, there are the things that could$_\blacklozenge$ have been its members. Formally:

(plural-comp*) $$\Box^*\forall y\Box^*\exists xx\forall z(z \prec xx \leftrightarrow z \in x \wedge @\blacklozenge(z \in y))$$

Let T be the theory comprising these four axioms. Then we can show in T that the following two claims are inconsistent. The first is the collapse$^\Diamond$ principle modified to take account of the actuality operator.[53]

(collapse$^\Diamond_@$) $$@\Box\forall xx@\Diamond(Exx \wedge \exists x(x \equiv xx))$$

The second claim is that we can$_\blacklozenge$ add a subset to some possible$_\Diamond$ set. Formally:

(add subsets) $$@\Diamond\exists x@\blacklozenge\exists y(y \subseteq x \wedge \neg@\Diamond Ey)$$

[53]The only slightly strange feature of this formulation is the first conjunct: Exx. We add this because, in the free logical setting, it could be that the xxs don't exist. And if we further assume, as may be plausible, that when the xxs don't exist they don't comprise anything, then in those circumstances the empty set would be co-extensive with them, thus trivialising collapse. See Roberts [ms] for discussion.

Theorem 2 In T, $\mathsf{collapse}^{\Diamond}_{@}$ and $\mathsf{add\ subsets}$ are jointly inconsistent.

In other words, $\mathsf{T} + \mathsf{collapse}^{\Diamond}_{@}$ proves that the possible$_{\Diamond}$ sets contain all their possible$_{\blacklozenge}$ subsets, formally:

$$@\Box\forall x @\blacksquare\forall y(y \subseteq x \to @\Diamond Ey)$$

3.2 Significance of the core result

As I mentioned above, it is an immediate consequence of the core result that the possible$_{\Diamond}$ sets do not constitute a universe of sets according to the width potentialist. For if they did, the width potentialist would have to claim that there are possible universes in which there are, for example, more subsets of the natural numbers. They would have to accept $\mathsf{add\ subsets}$.

As we've seen, most height potentialists assume that the possible$_{\Diamond}$ sets satisfy the axioms of ZFC. If they are right, then the possible$_{\Diamond}$ sets look for all intents and purposes like a universe of sets. In fact, they look like an ultimate universe of sets, an ultimate V. They are uniformly organised into the V_α hierarchy and by the core result they contain all of their possible$_{\blacklozenge}$ subsets. So, they contain every possible$_{\blacklozenge}$ subset of any possible$_{\Diamond}$ V_α. Indeed, we can show by induction on the possible$_{\Diamond}$ ordinals that when α is possible$_{\Diamond}$, any possible$_{\blacklozenge}$ V_α is a subset of the possible$_{\Diamond}$ V_α. In other words, the possible$_{\Diamond}$ V_αs subsume the possible$_{\blacklozenge}$ V_αs, when α is possible$_{\Diamond}$. This seems to show that *up to the height* of the possible$_{\Diamond}$ sets—which is to say, up to the height of the possible$_{\Diamond}$ ordinals—they comprise an ultimate V. In particular, since the possible$_{\Diamond}$ sets contain all possible$_{\blacklozenge}$ subsets of the natural numbers and all possible$_{\blacklozenge}$ well-founded countable ordinals, CH seems to get its ultimate and unambiguous formulation as $\mathsf{CH}^{\Diamond}$.

We can go further. It was a consequence of the width potentialist's central claim that any possible$_{\blacklozenge}$ set could$_{\blacklozenge}$ have been countable. Given the core result, this implies that any possible$_{\blacklozenge}$ well-founded set is possible$_{\Diamond}$.[54] So the possible$_{\Diamond}$ sets comprise absolutely all possible$_{\blacklozenge}$ well-founded sets.

Can the height potentialist weaken the assumption that the possible$_{\Diamond}$ sets satisfy the axioms of ZFC? As I argued in section 2, their central argument for $\mathsf{collapse}^{\Diamond}$ generalises to an argument that the possible$_{\Diamond}$ sets satisfy the axioms of $\mathsf{Z} + \forall x\exists\alpha(x \in V_\alpha)$. As above, the possible$_{\Diamond}$ V_αs will still subsume the possible$_{\blacklozenge}$ V_αs and contain any possible$_{\blacklozenge}$ subset of any possible$_{\Diamond}$ V_α, when α is possible$_{\Diamond}$. They will, in particular, contain all the possible$_{\blacklozenge}$ subsets of the natural numbers and all the possible$_{\blacklozenge}$ well-founded countable ordinals. Since they will still be uniformly organised into the V_α hierarchy, it again looks like *up to their height*, they comprise an ultimate V.[55]

Before we move on, let me mention two final related upshots of the core result. Suppose the width potentialist denies that the possible$_{\Diamond}$ sets constitute a universe of sets. (What the above observations show is that this would be to give up on their response to the problem

[54] The reason is that any such set can$_{\blacklozenge}$ be coded as a subset of ω. The core result then implies that the code possibly$_{\Diamond}$ exists and the set can$_{\Diamond}$ then be de-coded using the Mostowski collapse lemma. Indeed, this argument shows that every isomorphism type of a possible$_{\blacklozenge}$ set is possibly$_{\Diamond}$ realised, whether it is well-founded or not.

[55] It would not do for the width potentialist to claim that what possible$_{\Diamond}$ sets there are varies with the world$_{\blacklozenge}$ of evaluation. Perhaps, they might claim, we can't make good sense of the actuality operator: in general, that there's no absolute sense of what's possible$_{\Diamond}$. But the core result can be generalised to show that up to their height, the possible$_{\Diamond}$ sets from the perspective of *any* world$_{\blacklozenge}$ of evaluation constitute an ultimate V. Up to their heights, they will all contain precisely the same sets.

of independence. If the height potentialist is right, that problem arises with full force for the possible$_\Diamond$ sets.) It will still follow from the core result that the possible$_\Diamond$ powerset of the natural numbers contains all possible$_\blacklozenge$ subsets of the natural numbers. So whatever the width potentialist means by "universe of sets", the possible$_\Diamond$ powerset of the natural numbers cannot be an element of one. Similarly, since every possible$_\blacklozenge$ set is possibly$_\blacklozenge$ countable according to the width potentialist, it will follow that any possible$_\Diamond$ uncountable ordinal cannot be an element any universe of sets. Width potentialism would the concern only some rather than absolutely all sets. This limits its interest.

Furthermore, the width potentialist is faced with a difficult explanatory challenge. If the possible$_\Diamond$ sets satisfy the axioms of ZFC, then we can define the canonical intensional model for any possible$_\Diamond$ partial order. We can thus specify the corresponding range of ways the possible$_\Diamond$ sets could be, just as the width potentialist does within a universe of sets. But, by the core result, this specification cannot succeed in general: for various possible$_\Diamond$ partial orders, there can be no possible sets, in any sense, that are one of the ways so specified. Indeed, this is even true when we merely assume that the possible$_\Diamond$ sets satisfy the axioms of $\mathsf{Z} + \forall x \exists \alpha (x \in V_\alpha)$ since that theory can be used to define its own canonical intensional models which specify ways the possible$_\Diamond$ sets could be in which there are more subsets.[56] So, what is the *relevant* difference between the possible$_\Diamond$ sets and a universe of sets according to the width potentialist? Why can one be used to successfully specify alternative ways things could be but not the other?[57]

3.2.1 Actualism and width potentialism

Although we've viewed the core result as concerning height and width potentialism, it is ultimately a formal result that holds for any interpretation of the modal operators.

One particularly interesting interpretation arises when we take $\Diamond$ to mean something like "it is true in some stage of the iterative hierarchy that"—that is "it is true in some V_α that"—and @ to mean "it is true in the very first stage of the iterative hierarchy that"—that is, "it is true in V_0 that". As it was spelled out in section 2, the iterative conception implies that some things form a set just in case they all co-exist at some stage, at some V_α. Given this, it is straightforward to show that $\mathsf{collapse}^{\Diamond}_{@}$ holds on this interpretation. So, given the other axioms of T, it follows from the core result that it is impossible$_\blacklozenge$ to add subsets to sets in the iterative hierarchy. Actualism supplemented with the iterative conception is therefore just as incompatible with width potentialism as height potentialism.

This might strike you as strange. After all, although actualism and height potentialism aren't exhaustive, they seem to be the only plausible options in the vicinity. So width po-

[56] [details]

[57] The core result also shows that the standard models for combining height and width potentialism are inadequate. For example, as proposed in Hamkins and Linnebo [forthcoming] and Scambler [forthcoming], we can model the combination of these principles in the following way. We start with a countable transitive model M and consider the collection of all its forcing extensions $\mathcal{M}$. Our worlds would then be the $V_{\alpha+1}$s in these models—$W = \{V^{M'}_{\alpha+1} : M' \in \mathcal{M}\}$—where $V^{M'}_\alpha$ represents the first-order domain at a world and $V^{M'}_{\alpha+1}$ its plural domain, and where one world accesses another just in case the first is contained in the latter—$V^{M'}_{\alpha+1}$ accesses $V^{M''}_{\alpha+1}$ iff $V^{M'}_{\alpha+1} \subseteq V^{M''}_{\alpha+1}$. Then, we could interpret $\Diamond$ and $\blacklozenge$ in precisely the same way and @ could denote $V_1 = \{\emptyset\}$. It will follow that both $\mathsf{collapse}^{\Diamond}_{@}$ and $\mathsf{add\ subsets}$ hold in this model. But it can be shown that $\mathsf{plural\ comp}$ fails for very simple instances. ? showed that in such models there are subsets of ω, x and y that cannot co-exist. But an instance of $\mathsf{plural\ comp}$ would give us a sub-plurality of $\omega \times 2$ co-extensive with $\{\langle z, n\rangle : (z \in x \wedge n = 0) \vee (z \in y \wedge n = 1)\}$. $\mathsf{collapse}^{\Diamond}_{@}$ will then deliver their set, from which x and y can be defined.

tentialism had better be consistent with one of them! What's going on? The issue is that both the actualist who adopts an iterative conception and the height potentialist are playing the same game: they're both trying to provide a satisfactory response to the explanatory challenge to say when some things form a set. The height potentialist thinks they do a better job overall, but they're both trying to find an informative and explanatory account. For the actualist, it's a matter of co-existence at a stage; for the height potentialist, it's a matter of possible$_{\Diamond}$ co-existence. In either case, it follows that any subplurality of a set forms a set. By inspecting the proof of the core result, it's easy to see that this restricted form of collapse is all we need to derive our contradiction.[58]

There may be other ways to respond to the challenge which deny that subpluralities of sets form sets. But its hard to see how one could explain why *these* things in x form a set but *those* things don't. It seems more plausible that the width potentialist must reject the challenge and with it the motivation for height potentialism. For them, it must be to some extent random what sets there are in a given universe and thus what pluralities form sets in that universe. Indeed, this is suggested by the analogy they sometimes draw with geometry.[59]

The inconsistency between height and width potentialism runs deeper than the core result suggests. That result is downstream from a more fundamental disagreement about whether the explanatory challenge requires an informative and explanatory response.

4 Rejecting the background assumptions

In this section, I'll look at whether we can resist the core result by rejecting one of its background assumptions. Since the logical assumptions are uncontroversial, that leaves the axioms of T. I will consider them in turn.

4.1 Set extensionality and rigidity

Sets are frequently taken to be *the* paradigm example of *extensional* entities: entities that are completely characterised by what elements they have. It is standard to formulate this as the following non-modal principle.[60]

(*) $$\forall x, y(\forall z(z \in x \leftrightarrow z \in y) \to x = y)$$

But the idea that sets are completely characterised by their elements also motivates the modal extensionality principle, set-ext. If x and y have the same elements across worlds—that is, if the antecedent of set-ext, $\Box^*\forall z(\Diamond^*(z \in x) \leftrightarrow \Diamond^*(z \in y))$, is true—then the only way they could differ is not in *what* elements they could have, but *when* they could have them. In other words, they would have to be characterised *intensionally* after all.

Although neither of these extensionality principles implies set-rigidity outright, it has been unanimously adopted by authors working on modal set theory.[61] Nonetheless, I now want

[58]Note, in particular, that plural-comp* delivers a subplurality of x and that's the only plurality we need to collapse for the argument to go through.

[59]See, for example, Hamkins [2012].

[60]Boolos (REF) claimed, for example, that this extensionality principle is as good a candidate for an analytic truth as anything.

[61]See, for example, Linnebo [2013],Linnebo [2018], Studd [2019], Parsons [1983b], Fine [1981], Reinhardt [1980].

to argue that given some plausible further assumptions, (*) does imply set-rigidity.[62] As a background modal logic, I will use a positive free version of K for $\Diamond$.

The argument rests on two assumptions. The first assumption is that the possible$_\Diamond$ sets are closed under set subtraction. In particular, given any possible$_\Diamond$ sets x and y, there is a set which is exactly like x except that it does not contain y: it is x minus y. Let $x - \{y\}$ denote this set. By definition, it should have two properties. First, it should be necessarily co-extensive with x minus y. Formally:

$$\Box\forall z(z \in x - \{y\} \leftrightarrow z \in x \wedge z \neq y)$$

Next, it should not contain y. Formally:

$$y \notin x - \{y\}$$

The second assumption is that $x - \{y\}$ is compossible with x in various ways. In particular, if x could exist without y, that fact would be compossible with the existence of $x - \{y\}$. Formally:

$$\Diamond(Ex \wedge (\neg Ey \vee y \notin x)) \rightarrow \Diamond(Ex \wedge (\neg Ey \vee y \notin x) \wedge Ex - \{y\})$$

Now we can present the argument. Suppose (*) and that set-rigidity fails. Formally:

$$y \in x \wedge \Diamond(Ex \wedge (\neg Ey \vee y \notin x))$$

By the first and second assumptions we get:

$$y \in x \wedge \Diamond(Ex \wedge (\neg Ey \wedge y \notin x) \wedge Ex - \{y\} \wedge \forall z(z \in x - \{y\} \leftrightarrow z \in x \wedge z \neq y))$$

Since in the possible scenario y either fails to exist or to be an element of x, it follows that x and $x - \{y\}$ are co-extensive there. Since they both exist in that circumstance, (*) implies that they are identical there. Formally:

$$y \in x \wedge \Diamond(x = x - \{y\})$$

But by the definition of $x - \{y\}$, it doesn't contain y. That is:

$$y \in x \wedge y \notin x - \{y\} \wedge \Diamond(x = x - \{y\})$$

which contradicts the necessity of distinctness.[63]

Let me note two features of this argument. First, if we had assumed not that $y \in x$ but that $\Diamond(y \in x)$, then we would have derived:

$$\Diamond(y \in x \wedge y \notin x - \{y\}) \wedge \Diamond(x = x - \{y\})$$

[62] Of course, the upshot of this argument might be taken to be that in a modal setting we *shouldn't* automatically adopt (*): that, unlike set-ext, it does not, in the modal setting, follow from the claim that sets are completely characterised by their elements.

[63] This argument is similar to the argument from uniform adjunction for pluralities given in Linnebo [2016]. The current argument, however, does not rely on the B axiom. This is crucial, as in section 4.3 I will use this argument when $\Box$ is interpreted as a determinacy operator, which may not satisfy that axiom. [REF] Moreover,...

at the final line. But this is also inconsistent with the necessities of identity and distinctness, as they jointly imply:[64]

$$\Diamond(x = y) \to \Box(x = y)$$

So in addition to set-rigidity, the argument also establishes:

(set-rigidity*) $$\Diamond(x \in y) \to \Box(Ey \to (Ex \land x \in y))$$

The second feature of the argument I want to note is that in order to obtain set-rigidity or set-rigidity* for a modality $\Diamond$, it suffices that there is a broader modality—that is, a modality $\blacklozenge$ for which $\blacksquare\phi$ implies $\Box\phi$ for any ϕ—which satisfies the assumptions together with (*). It's hard to see how those assumptions would fail for $\Diamond^*$, given a height potential interpretation of $\Diamond$ and a width potential interpretation of $\blacklozenge$, but it's much harder to see how the assumptions must fail for *any* broader modality. And that's all we need.

4.2 Plural rigidity

Pluralities are also taken to be a paradigm case of extensional entities, and a similar argument could be given from the plural versions of (*) to plural-rigidity. However, there is a more direct route. We can justify plural-rigidity from the natural conception of pluralities and nothing over and above the things they comprise.

As Roberts [ms] argues, if some things are nothing over and above the things they comprise, and they comprise at least x, then they couldn't have existed without x existing nor without comprising at least x. There is no metaphysical gap between them and x: wherever they go, x must go too. And this is precisely what plural-rigidity says. To deny plural-rigidity is thus to deny this conception of pluralities.[65]

4.3 Plural comprehension

Earlier, I motivated plural comprehension using the conception of pluralities as nothing over and above the things they comprise. The idea was that if each individual ϕ exists, then nothing more is needed for there to be some things which are all and only the ϕs. The plurality of ϕs is nothing over and above the individual ϕs: they exist, taken together, if each of them exists individually. But this may have been too quick. The conception appears to be perfectly consistent with the thought that pluralities are *determinate*. But as Yablo [2006] has pointed out, this thought seems to be incompatible with certain instances of plural

[64] Indeed, in the presence of the T axiom, their conjunction is equivalent to this principle. And it, in turn, is equivalent to the claim that identity is non-contingent: formally, $\neg(\Diamond(x = y) \land \Diamond(x \neq y))$.

[65] Some have denied plural-rigidity and with it the nothing over and above conception of pluralities. See, for example, Hewitt [2012]. The argument may still have force even if the conception fails in general. To get the inconsistency, we just need some collection of pluralities—the nothing-over-and-above pluralities, say—that obey the principles. And most theorists who deny plural-rigidity in complete generality nonetheless think there is a rich subclass of pluralities for which it holds. [REFs]

comprehension. As he says:[66,67]

> The view once again is that plurality comprehension is mistaken.
>
> This may seem at first puzzling. The property P that (I say) fails to define a plurality can be a perfectly determinate one; for any object x, it is a determinate matter whether x has P or lacks it. How then can it fail to be a determinate matter what are all the things that have P? I see only one answer to this. Determinacy of the Ps follows from
>
> (i) determinacy of P in connection with particular candidates,
>
> (ii) determinacy of the pool of candidates.
>
> If the difficulty is not with (i), it must be with (ii). It is not the case that there are some things the [xxs] such that every candidate for being P is among them.

[66]It's worth pinpointing exactly where the incompatibility lies. Let Δ express determinacy. It is natural to think that the "is one of" relation is determinate, which we can formulate as the claim that if it$_x$ is one of them$_{xx}$, then it is determinately the case that it$_x$ is one of them$_{xx}$ (whenever they exist). This is really just one part of plural-rigidity, but for determinacy. Formally, the analogue of plural-rigidity for determinacy would be:

(*) $$x \prec xx \to \Delta(Exx \to x \prec xx \wedge Ex)$$

Indeed, we might even think that the considerations in favour of plural-rigidity extend to (*). Similarly, we might think that it should be determinate that if these$_{xx}$ and those$_{yy}$ comprise the same things, then they are the same. Formally:

(**) $$\Delta\forall x(\neg\Delta\neg(x \prec xx) \leftrightarrow \neg\Delta\neg(x \prec yy)) \to xx = yy)$$

These principles are perfectly consistent with the determinacy of plural comp, even when ϕ has indeterminate application conditions. Formally, they are consistent with:

(***) $$\Delta\exists xx\forall x(x \prec xx \leftrightarrow \phi)$$

when $\Delta\exists x(\neg\Delta\phi \wedge \neg\Delta\neg\phi)$. This can be seen by considering standard supervaluational models (e.g. REF). What *is* inconsistent with (*) and (**) (given an appropriate modal logic for Δ) is the following strengthened version of (*).

(****) $$\exists xx\Delta\forall x(x \prec xx \leftrightarrow \phi)$$

where ϕ has indeterminate application conditions. For Yablo's observations to have any force against (***) we need a notion of determinacy that supports the move from (***) to (****). In general, we need a notion which supports the move from $\Delta\exists xx\phi$ to $\exists xx\Delta\phi$. However, depending on our background assumptions, it is not at all obvious that the conception *is* compatible with this inference. For example, it—and its analogue for first-order quantifiers—is precisely the move that a proponent of classical logic must reject, since $\Delta\exists xx\phi$ will be true but $\exists xx\Delta\phi$ false when ϕ says that the xx are a cut-off point for a suitable sorties series. For example, there will be such a series for the plural predicate "surround the house". If we give up classical logic, then the thought becomes clearer. For example, suppose we think that in borderline cases the law of excluded middle fails. Then it seems that the conception is consistent with the claim that being one of is never borderline: formally, $x \prec xx \vee x \not\prec xx$. An instance of plural comprehension for ϕ would then imply that there are no borderline cases of ϕ, which will fail for indeterminate ϕ.

[67]Florio and Linnebo (2020) also deny plural comprehension. [details]

> If there were, one could go through the $[xx]$s one by one, asking of each whether it has P, thus arriving finally at the sought-after plurality of Ps.

The idea is that plural comprehension can fail either because its first-order quantifier is indeterminate or because the condition being comprehended on is indeterminate. To see how this affects the central result, let's return to the crucial instance of plural comprehension it employs.

$$\exists xx \forall z(z \prec xx \leftrightarrow @\blacklozenge(z \in y))$$

where y is a possible$_{\blacklozenge}$ subset of x. If Yablo is right, we could reject this instance either because "$\forall z$" is indeterminate or because the condition $@\blacklozenge(z \in y)$ is indeterminate. Let's consider these in turn.

Let Δ express the relevant notion of determinacy. The claim that $@\blacklozenge(z \in y)$ is indeterminate can be formalised as:

$$\neg\Delta@\blacklozenge(z \in y) \wedge \neg\Delta\neg@\blacklozenge(z \in y)$$

for some z. As is standard, I will assume that Δ validates the modal logic K. It follows that $\Delta@\blacksquare$ obeys the modal logic K. I claim that the assumptions of the argument in section 4.1 are highly plausible for this compound modal operator. After all, the extensionality principle (*) is something like an analytic truth; the first two assumptions are true in virtue of the definition of subtraction; and it's hard to see how we'd deny the compossibility assumption for this modal operator, given that we accepted it for $@\blacksquare$. Moreover, as I noted in section 4.1, these assumptions only need to hold for *some* modality broader than $\Delta@\blacksquare$. So, I will assume that it satisfies set-rigidity*. Formally:

$$\neg\Delta\neg@\blacklozenge(x \in y) \rightarrow \Delta@\blacksquare(Ey \rightarrow (Ex \wedge x \in y))$$

Given the indeterminacy of $@\blacklozenge(z \in y)$, it follows immediately that:

$$\neg\Delta@\blacklozenge Ey$$

In other words, it follows that it is not determinate that y possibly$_{\blacklozenge}$ exists. But that y possibly$_{\blacklozenge}$ exists is a *presupposition* of the core result. It would hardly change the upshot of the core result if we restricted it to determinately possible$_{\blacklozenge}$ subsets of the possible$_{\lozenge}$ sets.

What about the pool of candidates? Is the quantifier "$\forall z$" that we use to characterise the relevant plurality determinate? Note that by set-rigidity, the things in our target plurality are all in x whenever it exists. The proof would thus go through with the the following weaker instance of plural-comp*.

$$\exists xx \forall z(z \prec xx \leftrightarrow z \in x \wedge @\blacklozenge(z \in y))$$

But that instance is equivalent to:

$$\exists xx[\forall z(z \prec xx \rightarrow z \in x) \wedge \forall z \in x(z \prec xx \leftrightarrow @\blacklozenge(z \in y))]$$

Our target plurality, in other words, can be characterised merely by quantifying over elements of x. Assuming that the elements of a given set constitute a determinate totality, this means the range of our quantifier "$\forall z$" *can* be taken to be determinate.[68]

Before I end, let me note that in a sense the central result doesn't really depend on plural resources at all. The question when pluralities form (or rather could$_\Diamond$ form) sets is an instance of a much broader and more fundamental schematic question: when do (or can$_\Diamond$) conditions determine sets. The height potentialist's answer to this broader question follows from their answer to the narrower question: when the things satisfying the condition could have formed a plurality. But when pluralities behave as the nothing-over-and-above conception says, this response is equivalent to: when the things satisfying the condition could all have co-existed. The height potentialist's answer to the broader question is at bottom that a condition determines a set precisely when the things satisfying it could have all co-existed.[69] The actualist who adopts an iterative conception gives the alternative answer: when the things satisfying the condition could have all co-existed *at a stage*. The height potentialist's central charge is that by dropping the qualification to stages, we get a more explanatory account of sets.

So, given the nothing-over-and-above conception, pluralities provide a nice way to express the height potentialist's central claim, but they aren't necessary. We could equally well formulate that claim using backtracking operators, $\uparrow$ and $\downarrow$, as follows.[70]

$$(\mathsf{collapse}^{\downarrow}) \qquad \Box \uparrow \Diamond \exists x \forall y (y \in x \leftrightarrow \downarrow Ey \wedge \phi)$$

This new collapse principle is just as incompatible with width potentialism and the core result would go through if we replace plural-rigidity, plural comp, and collapse$^\Diamond$ with collapse$^\downarrow$.[71]

Whatever we think about pluralities, to give up collapse$^\downarrow$ is to give up on the height potentialist's answer to the central question and to give up on the supposed benefits of that answer over the actualist's.

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[68] All authors agree that the elements of a set *do* constitute a determinate totality. [REFs]

[69] To be precise: when they could have all co-existed *and* the condition is rigid. See Roberts [2016] for discussion.

[70] [REF for backtracking operators][comparison with Studd's operators]

[71] Just the instance of collapse$^\downarrow$ with $\phi = @\blacklozenge(z \in y)$.

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