Ultimate V

Draft

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1 Introduction

Potentialism is the view that the universe of sets is in some sense inherently potential. It comes in two main flavours. *Height potentialism* is based on the idea that sets are potential relative to their elements: once the elements exist, the set *can* exist. Take some people: Nadia, Dylan, and Melesha. Since each of them exists, the height potentialist claims that there could have been a set of them, that the set {Nadia, Dylan, Melesha} could have existed. Once we have that set, we can repeat the process. Given Nadia, Dylan, and Melesha and their set, the height potentialist will claim that there could have been a set of *those* things, that the set {Nadia, Dylan, Melesha, {Nadia, Dylan, Melesha}} could have existed. Continuing in this way, we get the possibility of more and more sets. So many, according to the height potentialist, that the sets thus obtained satisfy the axioms of set theory.[1]

Height potentialism is significant because it supports a compelling response to *Russell's paradox*. At its core, Russell's paradox tells us that some conditions fail to determine sets. In particular, that the condition of being a non-self-membered set fails to determine a set.[2] It thus raises a challenge: to provide an account of the dividing line between the conditions that do and do not determine sets that avoids problematic cases like the set of non-self-membered sets whilst delivering enough sets for the purposes of mathematics. The account offered by the height potentialist does this extremely well. It tells us that conditions determine sets precisely when all of their instances can co-exist: that set existence, in other words, is a matter of co-existence. With the right background assumptions, this basic idea can then be used to explain why there couldn't have been the problematic sets but why there can be enough sets for all of modern set theory.[3]

Width potentialism is based on the idea that some universes of sets are potential relative to others. Take an arbitrary universe of sets: $\mathcal{U}$. The width potentialist claims that by applying the *method of forcing* within $\mathcal{U}$, we can specify other possible universes of sets: universes, for example, in which there are more subsets of the natural numbers than there are in $\mathcal{U}$. Once $\mathcal{U}$ exists, those universes *can* exist. According to this view, no universe of sets is privileged. Every possible universe whatsoever can be used to specify further, richer, possible universes. There is no ultimate background universe of sets containing absolutely all sets or even absolutely all subsets of the natural numbers. There is no *ultimate* V. Rather, there is

[1] The labels "height potentialism" and "width potentialism" are sometimes used for a broader class of views than I will consider in this paper. See, for example, Hamkins and Linnebo [forthcoming]. But nothing important will rest on this terminological choice.

[2] Such a set would have to be a member of itself just in case it was not a member of itself!

[3] Height potentialism is most clearly expressed by recent writers like Linnebo [2010], Linnebo [2013], Studd [2019], Parsons [1977], and Hellman [1989]. But the view goes back at least to Putnam [1967] and Zermelo [1930], and can arguably be found even in Cantor (see Linnebo [2013] for discussion).

a broad space of equally legitimate universes, containing different sets and making different claims true.

Width potentialism is significant because it supports a compelling response to the *problem of independence*.[4] One of the central results of modern set theory is that its accepted axioms leave open some of its fundamental questions. The most famous example concerns the continuum hypothesis (CH), which says that every set of real numbers is either countable or has the same cardinality as the set of all real numbers. CH is neither provable nor disprovable from the currently accepted axioms of set theory. And despite significant efforts, set-theorists and philosophers have failed to find well-motivated principles that might prove or disprove it. Width potentialism explains with failure extremely well. According to the view, attempts to settle questions like whether CH is true or not are misplaced. CH is not an unambiguous statement for which we can marshal evidence. Rather, it is true or false only relative to a universe of sets. And in the broad space of possible universes of sets, we already know how CH behaves: how it is true in some universes and false in others. There is no ultimate V in which CH either unambiguously holds or fails to hold.[5]

It is natural to think that height and width potentialism are just aspects of a broader phenomenon of potentialism, that they might both be true.[6] The main result of this paper is that this is mistaken: height and width potentialism are jointly inconsistent. Indeed, I will argue that height potentialism is independently committed to an ultimate background universe of sets, an ultimate V, *up to its height*.

Here's the plan. In sections 2 and 3 I will outline height and width potentialism and motivate some of their basic features. In section 4 I use these features to show that the height potential sets constitute an ultimate V (up to their height) and thus that height and width potentialism are inconsistent. Section 5 considers some responses and section 6 is a technical appendix.

2 Height potentialism

Russell's paradox tells that there is no set of all the non-self-membered sets; no *Russell set*, r, for which

$$\forall x(x \in r \leftrightarrow x \notin x)$$

Some conditions, like "$x \notin x$", don't determine sets. Nevertheless, set theory tells us that many conditions do determine sets. The axiom of pairing, for example, tells us that the condition "$x = a \vee x = b$" determines a set whenever a and b are sets. We are thus faced with a challenge: to provide an account of the dividing line between the conditions that determine sets and those that don't which explains why there are enough sets for the purposes of set theory but not problematic sets like the Russell set.

The two standard responses to this challenge are based on the *limitation of size* and *iterative* conceptions of set. According to the limitation of size conception, set existence is a matter of size: conditions determine sets precisely when their instances are not too many. In particular, when they are fewer than the (von Neumann) ordinals. Properly formulated, the

[4]See Koellner [2006].

[5]Width potentialism finds its clearest formulation and defence in Hamkins [2012]. See also Hamkins and Linnebo [forthcoming].

[6]See, for example, Hamkins and Linnebo [forthcoming] and Scambler [forthcoming].

limitation of size conception explains why many of the axioms of set theory are true whilst also explaining why there is no Russell set.[7] The non-self-membered sets are not fewer than the ordinals, because every ordinal *is* a non-self-membered set.

According to the iterative conception, the sets occur in a well-ordered series of stages. At the very first stage, we have no sets whatsoever. Then, at the second stage, we have all the sets of things at the first stage. Since there's nothing at the first stage, that means the only set at the second stage is the empty set: $\emptyset$. At the third stage, we have all the sets of things at the second stage. Since the empty set is the only thing at the second stage, that means the sets at the third stage are the set containing the empty set together with the empty set itself: $\{\emptyset\}$ and $\emptyset$. At the fourth stage, we have all the sets of *those* things. And so on. In general, at any given stage we have all and only the sets of things available at some previous stage. On this conception, conditions determine sets precisely when their instances all occur at some stage. Set existence, in other words, is a matter of co-existence at a stage. Properly formulated, the iterative conception also explains why many of the axioms of set theory are true whilst explaining why there is no Russell set.[8] The non-self-membered sets do not all co-exist at any one stage because no matter what stage we consider, there will always be a non-self-membered set outside of it.[9]

Both conceptions licence a schema of separation, which says that for any set x, there exists a set of all and only the ϕs in x.

(Separation) $$\forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge \phi)$$

On the limitation of size conception, the members of any set are fewer than the ordinals. So, trivially, the ϕs among them are also fewer than the ordinals and thus form a set. On the iterative conception, the members of any set all co-exist at some stage. So, trivially, the ϕs among them all co-exist at that very same stage and thus form a set. These arguments, moreover, aren't sensitive to the specific features of ϕ: they hold good for any (meaningful) ϕ whatsoever. The schema is *open-ended*. To reject any instance of Separation in any language is to give up on the simple idea that set existence is merely a matter of size or co-existence at a stage.

Height potentialism offers an alternative response to the challenge posed by Russell's paradox. It is based on the idea that sets are *potential* relative to their elements: once the elements all exist together—once they all co-exist—the set *can* exist.[10] Assuming that sets cannot exist without their elements,[11] it follows that the set can exist precisely when the elements can all co-exist. What emerges, then, is a picture on which set existence is a matter of co-existence. Not co-existence at a stage, like the iterative conception, but co-existence simpliciter.

[7] In particular, we get the following axioms .... See Levy for union. [REF]

[8] In particular, we get the axioms of pairing, union, powerset, separation, and the claim that every set is in some stage of the iterative hierarchy: $\forall x \exists \alpha (x \in V_\alpha)$. [REF] Boolos The axioms of infinity and replacement can then also be obtained via a natural reflection principle. [REF] Paseau, though warning about Pseasu and partial vs complete.

[9] Namely, the set of non-self-membered sets from the given stage.

[10] A crucial question for the height potentialist is how they understand the relevant notion of potentiality, the sense in which sets *can* exist. For some, it is a distinctively mathematical notion (Linnebo [2013], Parsons [1983b], Hellman [1989], and Reinhardt [1980]); for some, an interpretational notion (Linnebo [2018], Studd [2019] and Uzquiano [2015]); and for others, a logical notion (Hellman [1989] and Berry [2018]). I will not take a stand on this issue here, since my arguments will by and large rest on assumptions that hold for all of these views. I will explicitly flag when they don't.

[11] See the Stability$^\Diamond$ principle in section 4 and the discussion in section 5.

It is now standard to make height potentialism precise using plural quantification,[12] given a conception of pluralities as nothing over and above the things they comprise.[13] As Roberts [forthcoming] shows, this conception breaks down into three component ideas: *upward dependence*, *downward dependence*, and *extensionality*. Upward dependence says that some things taken individually suffice for those very same things taken together. For example, consider our people: Nadia, Dylan, and Melesha. The idea is that if each of them individually exists, then *they* must exist. When you have the individual things, you have the plurality. An important consequence of upward dependence is the plural comprehension schema which says that there are some things comprising all and only the ϕs, for each condition ϕ.

(Plural Comp) $$\exists xx \forall x(x \prec xx \leftrightarrow \phi)$$

The thought is that since each individual ϕ exists and the ϕs taken together are nothing over and above the ϕs taken individually, they must exist. Like Separation, Plural Comp is open-ended. The argument from upward dependence holds good no matter what (meaningful) ϕ we consider.

Downward dependence is the converse of upward dependence. It says that some things taken together suffice for those very same things taken individually. Again, consider our people. The idea is that if *they* exist, then so too must Nadia, Dylan, and Melesha. When you have the plurality, you have the individual things. Extensionality simply says that pluralities comprising the same things are identical.

Together, upward and downward dependence intuitively ensure that co-existence is a matter of the existence of a plurality. Upward dependence ensures that when the possible ϕs all co-exist, they determine a plurality (by Plural Comp); and downward dependence ensures that when there could have been a plurality of all the possible ϕs, they could have all co-existed. Pluralities thus provide us with a precise and perspicuous way to talk about co-existence. In particular, it allows us to formulate the central height potentialist thought that sets are potential relative to the co-existence of their elements as the claim that necessarily any plurality could have formed a set.

(Collapse$^\Diamond$) $$\Box \forall xx \Diamond \exists x(x \equiv xx)$$

where $x \equiv xx$ abbreviates $\forall y(y \in x \leftrightarrow y \prec xx)$.

With the right background assumptions,[14] Plural Comp and Collapse$^\Diamond$ can be used to show that the dividing line between conditions that can determine sets and those that can't is indeed possible co-existence (which is to say, the possible existence of a plurality). There is a caveat. Say that a condition is *stable* if it doesn't change its instances: if, for any possible object, when it's an instance, it's necessarily an instance; and when it isn't, it necessarily

[12]See, for example, Linnebo [2010], Linnebo [2013], and Studd [2019]. Studd [2013] presents one alternative formulation. Another would be to use so-called backtracking operators, ↑ and ↓. For example, we could formulate the claim that set existence is a matter of co-existence as the following schema (taking into account the caveat discussed below that ϕ must be possibly stable).

$$\Diamond \exists x \Box \forall y(y \in x \leftrightarrow \phi) \quad \leftrightarrow \quad \Diamond \uparrow \Box \forall y(\phi \rightarrow \downarrow Ex)$$

[13]For convenience, I will frequently use the singular "plurality" and talk of objects being elements or members of pluralities. Nothing rests on this, however, and everything I say can be reformulated using genuinely plural locutions.

[14]See, for example, Linnebo [2013] or Roberts [2016].

isn't. Formally: $\Box\forall x(\Box\phi \vee \Box\neg\phi)$. Then, what we can show is that a condition can determine a set precisely when its instances can all co-exist (which is to say, when there could have been a plurality of all its possible instances) *and* it's possibly stable.

$$\Diamond\exists y\Box\forall x(x \in y \leftrightarrow \phi) \quad \leftrightarrow \quad [\Diamond\exists xx\Box\forall x(\phi \to x \prec xx) \wedge \Diamond\Box\forall x(\Box\phi \vee \Box\neg\phi)]$$

This result can be used, in turn, to explain why many of the axioms of set theory are true in the potential sets,[15,16] whilst also explaining why there is no Russell set. There could not have been a plurality of all the possible non-self-membered sets because no matter what sets we consider, there could have been a non-self-membered set not among them.[17]

Just as the limitation of size and iterative conceptions license a non-modal schema of separation, height potentialism licences a modal schema of separation, which says that for any possibly stable condition ϕ and any possible set x, there could have been a set of all and only the ϕs in x.

(Separation$^\Diamond$) $$\Diamond\Box\forall z(\Box\phi \vee \Box\neg\phi) \quad \to \quad \Box\forall x\Diamond\exists y\forall z(z \in y \leftrightarrow z \in x \wedge \phi)$$

Intuitively, since the members of any set x can all co-exist (because x's members exist whenever it exists), it is trivial that the ϕs among them can all co-exist and thus could have formed a set. When ϕ is stable, the ϕs among x's members will necessarily be the ϕs among them. So, there could have been a set of the ϕs among x's members. This thought, moreover, isn't sensitive to the specific features of ϕ: it holds good for any (meaningful) ϕ whatsoever. So, just like Separation, Separation$^\Diamond$ is open-ended. To reject Separation$^\Diamond$ is to give up on the simple idea that set existence is merely a matter of co-existence. In more precise, plural, terms: the instance of Separation$^\Diamond$ for ϕ is implied by the corresponding instance of Plural Comp together with Collapse$^\Diamond$. Since Plural Comp is open-ended, so too is Separation$^\Diamond$. To reject Separation$^\Diamond$ is to give up either on the simple idea that plural existence is merely a matter of co-existence or to give up on the simple idea that any possible things could have formed a set. In either case, this would mean giving up on their response to Russell's paradox.[18]

The central argument in favour of height potentialism is that it does better than its competitors at answering the explanatory challenge raised by Russell's paradox.[19] In particular, the claim is that, in contrast to height potentialism, both the limitation of size and iterative conceptions fail to adequately explain why crucial conditions do or do not determine sets. For example, on pain of contradiction, we know that the ordinals do not form a set. According to the limitation of size conception:

> the explanation is that [the ordinals] are too many to form a set, where being too many is defined as being as many as [the ordinals]. Thus, the proposed explanation moves in a tiny circle. [Linnebo, 2010, p.154]

[15] We say that a statement in the language of set theory is true in the potential sets when its *modalisation* is true, where the modalisation of a statement ϕ is the result of replacing occurrences of $\exists x$ and $\forall x$ in ϕ it with $\Diamond\exists x$ and $\Box\forall x$ respectively. This has the effect of forcing its quantifiers to range over all potential sets. See Studd [2019], Linnebo [2010], and Linnebo [2013].

[16] In particular, we get pairing, union, and separation. See [REF] for details. The axioms of foundation, infinity, powerset, and replacement require further assumptions. See footnote ? for discussion.

[17] Namely, the set of non-self-membered sets among them.

[18] Either because they then have nothing to say about set existence—without Collapse$^\Diamond$—or because they have nothing to say about plural existence—without the nothing over and above conception.

[19] See, for example, Linnebo [2010] and Studd [forthcoming].

In contrast, the height potentialist offers a non-circular explanation: the ordinals do not form a set because they cannot all co-exist. Similar problems are raised for the iterative conception.[20] Co-existence is thus claimed to be a more explanatory notion than both size or co-existence at a stage. Other things being equal, that's a reason to prefer height potentialism.

As it stands, though, height potentialism doesn't motivate as many principles of set theory as the limitation of size or iterative conceptions. For example, it fails to motivate the axioms of foundation and powerset, both of which are easy consequences of the iterative conception. Similarly, it fails to motivate the claim that every set is in some rank in the cumulative hierarchy: formally, $\forall x \exists \alpha (x \in V_\alpha)$.[21] In many ways, this claim is the *core* assumption of modern set theory.[22] It is a consequence of ZF and, given the axioms of separation and extensionality, it implies all of the other axioms of ZF with the exception of infinity and replacement. Those axioms, in turn, are naturally seen as assumptions about how far the ranks extend, which is to say, how far the ordinals extend. The axiom of infinity, for example, tells us that the ordinals extend so far that there is at least one infinite ordinal and the axiom of replacement tells us that the ordinals extend further than any ordinals that can be put in one-one correspondence with the members of a set. Similarly, it is natural to see large cardinal hypotheses—which have been suggested as additions to ZF—as claims about the extent of the ranks. Most of the set theories currently taken to be well-motivated by set theorists can thus be seen as a combination of the core claim that every set is in some rank together with some claim (or claims) about how far the ordinals extend.

It is therefore incumbent on the height potentialist to supplement their account, so that their explanatory successes aren't overshadowed by a loss of strength. And indeed, this is precisely what they do. Linnebo [2013] and Studd [2019], for example, both supplement the above principles in such a way as to prove that all of the theorems of ZF are true in the

[20] I find these, however, much less worrisome. For example, Linnebo [2010] argues that the unexplanatoriness of the limitation of size conception is inherited by the iterative conception. He shows that given plausible assumptions, the iterative conception *implies* the limitation of size claim that conditions determine sets precisely when their instances are fewer than the ordinals. The problem with this argument is that although the proponent of the iterative conception may be committed to such a claim, they need not think it carries any *explanatory* weight. It may be that Zara is fond of all and only the pluralities that form sets. But it doesn't follow that Zara's fondness for a plurality explains why it does or does not form a set. Explanation is a highly intensional notion and equivalent statements will not always explain the same things. As should be clear from the above discussion, I find the most natural explanatory notion for the iterative conception to be co-existence at a stage. The ordinals do not form a set because they do not all co-exist at some stage. Which is not at all circular.

Studd [2019] focuses on a different example. For him, the most natural explanatory notion for the iterative conception is the bounding of ranks in the V_α hierarchy. The the finite ordinals *do* form a set because there is some ordinal greater than their ranks. But as each ordinal *is* its own rank, the proposed explanation is that they form a set because there is some ordinal greater than them. Which is to say: they form a set. As before, the proposed 'explanation moves in a tiny circle'. However, as above, I see no reason for the proponent of the iterative conception to take bounding of ranks, rather than co-existence at a stage, to be the relevant explanatory notion.

[21] The V_αs correspond to the stages of the iterative conception, but are explicitly definable within ZF by a simple recursion on the ordinals. We let V_0 be the empty set, $V_{\alpha+1}$ the powerset of V_α, and V_λ the union of V_α for $\alpha < \lambda$, when λ is a limit ordinal.

[22] As [Reinhardt, 1974, p. 190] puts it:

> ...a general and universal framework... [has now] been provided for set theory by the clarification of the intuitive idea of the cumulative hierarchy (due chiefly to Zermelo) (i.e., the sets in the series [V_0, V_1,...] ...). We might say this reduces all structural questions to questions about "arbitrary subset of" and "arbitrary ordinal".

potential sets and thus that $\forall x \exists \alpha (x \in V_\alpha)$ is true in the potential sets.[23] In proving the inconsistency of height and width potentialism in section 4, I'm going to rely on the fact that the height potentialist is committed to this latter claim. So, for what it's worth, I want to finish this section with an independent argument that the height potentialist should be committed to it.

Height potentialism is a view on which set existence is contingent.[24] For example, if rr are the non-self-membered sets—formally, if:

$$\forall x(x \prec rr \leftrightarrow x \notin x)$$

then their set, $\{rr\}$, could have existed, by $\mathsf{Collapse}^{\Diamond}$, but doesn't (on pain of contradiction). $\{rr\}$, in other words, is a merely contingent being. I take it, therefore, that the most natural way to reject height potentialism is to deny such contingency; to claim that any set which could exist *does* exist, that set existence is non-contingent. Call this view *height actualism*. Formally, it can be captured by the Barcan and Converse Barcan formulas. The Barcan formula says that if there could be a ϕ (alternatively, if there could be some things that are ϕ), then there is already something which could be ϕ (alternatively, there are some things which could be ϕ). Intuitively, it ensures that whatever could exist already exists. Formally:

(BF) $$\Diamond \exists \mathbf{x} \phi \rightarrow \exists \mathbf{x} \Diamond \phi$$

where $\mathbf{x}$ is either a first-order or plural variable. The Converse Barcan formula, on the other hand, says that if there is something which could be ϕ (alternatively, if there are some things which could be ϕ), then there could be something which is ϕ (alternatively, there could be some things which are ϕ). It ensures that whatever exists, must exist. Formally:

(CBF) $$\exists \mathbf{x} \Diamond \phi \rightarrow \Diamond \exists \mathbf{x} \phi$$

Together, then, BF and CBF effectively ensure that existence is non-contingent.[25]

Height actualism flattens the modal distinctions that height potentialism exploits. Take $\mathsf{Collapse}^{\Diamond}$, for example. We can use BF and CBF to prove that some things could have formed a set precisely when they do in fact form a set and so that $\mathsf{Collapse}^{\Diamond}$ is equivalent to the false claim that any things form a set.[26] Formally:

(Collapse) $$\forall xx \exists x (x \equiv xx)$$

[23] Studd [2019] obtains the powerset axiom by adopting a stronger version of collapse (see below), whereas Linnebo [2013] simply adds the powerset axiom as a further assumption. Similarly, Studd [2019] secures foundation via a natural modal principle, whereas Linnebo [2013] simply assumes it. Both adopt a modal *reflection principle* to obtain the axioms of infinity and replacement.

[24] Of course, contingent in the sense of '$\Box$', however the height potentialist spells that out.

[25] More precisely, we can obtain height actualism from height potentialism by dropping $\mathsf{Collapse}^{\Diamond}$ and adding BF and CBF. See Roberts [2016] for details. Thus formulated, height actualism is a view about possibility in the height potentialist's sense. An alternative—one that I'm sympathetic to—is to take height actualism to be an ideologically *quietist* position, one that simply resists adopting the height potentialist's modality.

[26] In fact, with the right background assumptions, we can prove that when we restrict our attention to claims solely about sets and pluralities, the modality is completely redundant. That is, we can prove:

$$\phi \leftrightarrow \Box \phi \leftrightarrow \Diamond \phi$$

See Roberts [2016].

Or, take the central question raised by Russell's paradox: when does a condition ϕ determine a set? In the modal setting, this was implicitly taken to be the question: when could there have been a set that necessarily contains all and only the ϕs? Formally, when is it the case that $\Diamond\exists x\Box\forall y(y \in x \leftrightarrow \phi)$? But we could have taken it to be the non-modal question: when is there a set of all and only the ϕs? Formally, when is it the case that $\exists x\forall y(y \in x \leftrightarrow \phi)$? These two questions are equally legitimate and give rise to their own explanatory challenges. The first, *inter-world* challenge, is the one I've focused on: namely, to draw an informative line between the conditions that determine sets *across worlds* and those that don't. The second, *intra-world* challenge, is to draw an informative line between the conditions that determine sets *within worlds* and those that don't.[27]

For the height actualist, the inter- and intra-world challenges are equivalent.[28] Set existence and possible set existence effectively come to the same thing, and both, we can assume, are a matter of size or co-existence at a stage. For the height potentialist, the challenges are crucially different, and although they have a response to the inter-world challenge, as it stands they have very little to say about the intra-world challenge. The dividing line between the conditions that do and do not determine sets at worlds varies wildly between models of $\mathsf{Collapse}^{\Diamond}$ and $\mathsf{Plural\ Comp}$ and indeed between worlds within a single model.[29]

It would do no good for the height potentialist to adopt the limitation of size or iterative conceptions in response to the intra-world challenge. To say, for example, that there is a set of ϕs at a world precisely when there is a stage at that world where all the ϕs occur. Because then the proposed explanations would be precisely the same as those offered by the height actualist. The height actualist would effectively face one challenge—since the inter- and

[27]The force of the intra-world challenge will of course depend on how we understand the relevant notion of potentiality. An analogy: suppose we want to know when seeds will grow into trees. Asked of biologically possible worlds, this seems like a perfectly legitimate question, demanding an explanatory answer. But asked of metaphysically possible worlds more generally, it doesn't. There's likely nothing informative we can say about when a seed will grow into a tree in metaphysical generality. The question is then whether the height potentialist's worlds are to pluralities and sets as biologically possible worlds are to seeds and trees. One reason to think they are is that some of the height potentialist's background assumptions require worlds to be mathematically well-behaved and law-like just like biologically possible worlds are biologically well-behaved and law-like. See, for example, the stability, inextensibility, and extensionality principles in (Linnebo [2013]).

Hellman [1989] is an interesting exception. For him, worlds are effectively *defined* to be logically possible models of second-order ZFC (see Roberts [2019] for details). Worlds are then well-behaved and law-like by definition. See Berry [2018] for a compelling modification of Hellman's account that identifies Hellman's logically possible models with the corresponding logically possible worlds.

[28]More precisely, by Plural Comp, there are some things xx that are all and only the ϕs. The height actualist can then use BF and CBF to prove that:

(i) $$\exists x\forall y(y \in x \leftrightarrow \phi) \quad\leftrightarrow\quad \exists x\forall y(y \in x \leftrightarrow y \prec xx) \quad\leftrightarrow\quad \Diamond\exists x\Box\forall y(y \in x \leftrightarrow y \prec xx)$$

An answer to the inter-world challenge is thus an answer to the intra-world challenge.

Now, when ϕ is stable, they can also use BF and CBF to prove that:

(ii) $$\Diamond\exists x\Box\forall y(y \in x \leftrightarrow \phi) \quad\leftrightarrow\quad \exists x\forall y(y \in x \leftrightarrow \phi)$$

Therefore, when ϕ is possibly stable, it is possible that $\Diamond\exists x\Box\forall y(y \in x \leftrightarrow \phi)$ is equivalent to $\exists x\forall y(y \in x \leftrightarrow \phi)$. Assuming what's possibly possible is possible, it follows that an answer to the intra-world question in this case will answer the inter-world question. On the other hand, when ϕ is necessarily unstable, $\Diamond\exists x\Box\forall y(y \in x \leftrightarrow \phi)$ cannot be true. See Roberts [2016].

[29]In particular, if C is any class of transitive sets for which (i) for any $x \in C$ and $y \subseteq x$, there is $z \in C$ such that $y \in z$ and (ii) for any $x, y \in C$ there is $z \in C$ such that $x \cup y \subseteq z$, then C will model Plural Comp and $\mathsf{Collapse}^{\Diamond}$—together with the height potentialist's other background assumptions—under the obvious interpretation.

intra-world challenges are equivalent for them—and give what the height potentialist thinks is a relatively unexplanatory response and the height potentialist would face two challenges, giving an equally unexplanatory response to one of them. A stalemate.

A version of the intra-world challenge arises for the iterative conception. We can ask: when does a condition determine a set at a stage? Luckily, as I formulated it above, it already embodies an answer.[30] First, it implies a principle of *priority* according to which a set occurs at a stage *only if* its elements are all available at some prior stage. This puts an upper bound on the sets available at any given stage: it tells us that they comprise *at most* sets of things available at some prior stage. Second, it implies a principle of *plenitude* according to which a set occurs at a stage *if* its elements were all available at some prior stage. This puts a lower bound on the sets available at any given stage: they comprise *at least* sets of things available at some prior stage. Together, the priority and plenitude principles imply that a condition determines a set at a stage precisely when its instances all co-exist at some prior stage. It's hard to see, moreover, what other, non-arbitrary, answer there could be.

For height potentialism, the analogue would be the claim that a condition determines a set at a world precisely when its instances all co-exist at some prior world. As with the iterative conception, it's hard to see what other, non-arbitrary, response to the intra-world challenge there could be. But to even express this claim, we need further modal resources. In particular, we need a modal operator that "looks back" to prior worlds in the same way that $\Diamond$ "looks forward" to subsequent worlds. Following Studd [forthcoming], we can do this by adding to the language of set theory not a single modal operator $\Diamond$, but a pair of operators $\Diamond^<$ and $\Diamond^>$ meaning roughly that it is the case at some subsequent world and that it is the case at some prior world respectively. $\Box^<$ and $\Box^>$ are then defined in the obvious way. Let $\boxdot$ be an operator expressing truth at the current world and all prior and subsequent worlds. Formally, let $\boxdot\phi$ abbreviate $\Box^<\phi \wedge \phi \wedge \Box^>\phi$. Then the priority and plenitude ideas can be formalised as follows.

(Priority$^\Diamond$) $$\boxdot\forall x\Diamond^>\exists xx(x \equiv xx)$$

(Plenitude$^\Diamond$) $$\boxdot\forall xx\Box^<\exists x(x \equiv xx)$$

Together, Priority$^\Diamond$ and Plenitude$^\Diamond$ imply that some things form a set at a given world precisely when they all co-exist at some prior world. Formally:

$$\boxdot\forall xx(\exists x(x \equiv xx) \leftrightarrow \Diamond^> Exx)$$

It is then an immediate consequence of Plural Comp that a condition determines a set at a world precisely when its instances all co-exist at some prior world. Since Plenitude$^\Diamond$ implies Collapse$^\Diamond$, we thus get a response to both the intra- and inter-world challenges.[31] As

[30] Other ways of formulating the conception do not. See, for example, the formulation proposed by Boolos [1971].

[31] There is a related, but much harder, challenge facing the height potentialist. Priority$^\Diamond$ and Plenitude$^\Diamond$ tell us that a condition determines a set at the actual world when its instances all co-exist at some prior world. But they leave wide open the question which worlds *are* prior to the actual world. The challenge is to provide an explanatory answer to this question, and the problem is that it's very hard to see how the height potentialist might do this. (There is, at least, one natural and non-arbitrary answer: namely, that the actual world is the very first world—that is, the world with no sets whatsoever by Priority$^\Diamond$. But even given this answer, it would still remain mysterious why the actual world is the first, why there are actually no sets.)

Studd [2019] shows, with the right background assumptions, it follows from Priority$^{\Diamond}$ and Plenitude$^{\Diamond}$ that the core claim that every set is in some rank in the cumulative hierarchy—$\forall x \exists \alpha (x \in V_\alpha)$—holds in the potential sets.

In summary: I've argued that the height potentialist is committed to two things. First, the open-ended modal separation principle, Separation$^{\Diamond}$. To give this up is to give up their response to Russell's paradox, which was the primary motivation for height potentialism. Second, the claim that in the potential sets, every set is in some rank of the cumulative hierarchy—$\forall x \exists \alpha (x \in V_\alpha)$. The challenge posed by Russell's paradox bifurcates in the modal setting into the intra- and inter-world challenges. Without Priority$^{\Diamond}$ and Plenitude$^{\Diamond}$, it's hard to see how the height potentialist can adequately respond to both. With them, it follows that $\forall x \exists \alpha (x \in V_\alpha)$ holds in the potential sets.

3 Width potentialism

Width potentialism is the view that, by applying the method of forcing within a given universe of sets, we can specify other possible universes of sets; universes, for example, in which there are more sets of natural numbers. The idea, in other words, is that some universes of sets are potential relative to others: once a given universe exists, universes specified by applying the method of forcing within it *can* exist. For now, it will be helpful to mostly ignore the potentialism part—to ignore the 'possible', 'potential', and 'can'—and simply take width potentialism to be the non-modal view that, via the method of forcing, universes of sets specify other universes of sets. I will return to the question what exactly universes of sets *are* later. Until then, I will simply assume that they are transitive collections of sets whose members satisfy the axioms of ZFC.[32]

To understand width potentialism, we need to look at the fundamentals of forcing. The basic idea is simple. It is standard in set theory to use the axioms of ZFC to construct models of those very same axioms. The most well-known example is Gödel's model, L, comprising the so-called constructible sets. In ZFC, we can prove both that each axiom of ZFC is true in L and that the continuum hypothesis (CH) is true in L. We conclude that ZFC does not prove the negation of CH, assuming it's consistent: if it did, it would prove that both CH and its negation are true in L, which is a contradiction. Think of L as specific way the universe of sets could be: the way it would be if every set were constructible. So the general strategy

One idea is to undercut the challenge by taking an "external perspective" on the height potential worlds. The thought is that the actual world need not be among the height potential worlds and so the question where it's located among them need not even arise. (Strictly speaking, this would mean giving up on the T principle of modal logic.) But that merely changes the formulation of the challenge, rather than its force. The challenge would then be to provide an explanatory answer to the very simple non-modal question: what sets are there? For which ϕ, in other words, is it the case that $\exists x \forall y (y \in x \leftrightarrow \phi)$? If Priority$^{\Diamond}$ and Plenitude$^{\Diamond}$ don't apply to the actual world, the height potentialist doesn't even have the beginnings of an answer. For the height actualist, the question is, again, effectively equivalent to the others we've considered. Without a response to this problem, the height potentialist may have to accept that the balance of explanatory virtue does not tip so heavily in their favour after all. See Menzel [REF] for a related point.

[32] More precisely, let $x \in \mathcal{U}$ mean that x is in the universe $\mathcal{U}$ and for ϕ in the language of set theory, let ϕ^U be the result of restricting its quantifiers to $\mathcal{U}$: so, $\phi^{\mathcal{U}}$ is the result of replacing occurrences of $\exists x$ in ϕ with $\exists x \in \mathcal{U}$. Then the idea is that for any universe $\mathcal{U}$ and any axiom ϕ of ZFC, $\phi^{\mathcal{U}}$ and for any sets x, y, if $x \in y$ and $y \in \mathcal{U}$, then $x \in \mathcal{U}$. In addition, I will assume that sets are extensional in the sense that sets with the same elements—no matter what universes those elements come from—are identical, and that set membership is well-founded.

is to find specific ways the universe of sets could be in which the axioms of ZFC are all true but in which some target claim is false, thereby showing that the target claim doesn't follow from those axioms.

Forcing generalises this strategy. Instead of helping us to find specific ways the universe of sets could be, however, it gives us a recipe for finding *ranges* of ways the universe of sets could be. More concretely, it tells us how to use the axioms of ZFC to construct *intensional* models in which those very same axioms are necessarily true, rather than true simpliciter, but in which some target claim—like CH—is possibly false. As above, we conclude that ZFC does not prove the target claim, assuming it's consistent: if it did, it would prove both that the target claim is necessarily true in the relevant intensional model—since that claim follows from the axioms of ZFC and each of those axioms is necessarily true—and that its negation is possibly false, which is a contradiction.

Since each universe of sets satisfies the axioms of ZFC, the method of forcing thus tells us how to construct these intensional models within them. Width potentialism goes beyond this by claiming that given a universe $\mathcal{U}$ and a suitable intensional model specifying a range of ways the universe of sets could be, there is some universe $\mathcal{U}'$ which is one of those ways. Let me make this precise.

In specifying a range of ways the universe could be, intensional models are like Kripke models. But whereas Kripke models deal with possible worlds, intensional models deal with *possibilities*. Possibilities are like *parts* of possible worlds. Where possible worlds are complete specifications of ways the world could be, possibilities are partial specifications. There is, for example, the possibility that Mae loves George. It is part of any possible world in which Mae loves George but it has little to no opinion about other matters. For example, it has no opinion about whether Mae works as a stand up comedian. Nevertheless, we can *extend* it to more inclusive possibilities that settle such questions: possibilities in which Mae loves George and works as a stand up, and possibilities in which Mae loves George and doesn't work as a stand up.[33]

A space of possibilities can be modelled by a *partial order*.[34] We can think of the elements of the partial order as the possibilities and its relation $\leq$ as inclusion between them. So, for possibilities p and q, the idea is that $p \leq q$ when p has q as a part; when p includes q; when p extends q. For example, q might be the possibility that Mae loves George and p the possibility that Mae loves George and works as a stand up.

Let $\mathbb{P}$ be any partial order. We can obtain an intensional model from $\mathbb{P}$ in two simple steps. First, we decide what the domain of the model will be. Since we're focusing on set theory, its elements will act as the possible sets of the model. Second, for our atomic relations, we decide which possible sets will relate to which relative to what possibilities. Again, since we're focusing on set theory, that means the relations of membership and identity. Formally, then, an intensional model is obtained by supplying a domain of objects D and two interpretation

[33]Possible worlds can be identified with maximally inclusive possibilities: possibilities that cannot be extended to more inclusive possibilities. Whether there are such possibilities will depend on how rich our space of possibilities is. For some spaces, it may be that all possibilities are extendable and thus that there are no possible worlds in this sense. When every possibility can be extended to a maximally inclusive possibility, the resulting intensional models effectively collapse to Kripke models. The interesting cases are thus those where this fails. Of course, since classical models are a special case of Kripke models, when there is a single possibility that extends every possibility, the resulting intensional models will effectively collapse to classical models.

[34]Which is to say a domain together with a reflexive, transitive, and anti-symmetric relation $\leq$ over that domain.

functions $I_\in$ and $I_=$ that map ordered pairs of elements of D to subsets of $\mathbb{P}$. Suppose, for example, that our interpretation function $I_\in$ assigns the set of all possibilities to a pair $\langle x, y\rangle$: that is, $I_\in(\langle x, y\rangle) = \mathbb{P}$. Then according to that assignment, the claim that x is a member of y is *necessary.* No matter how things had been, x would have been an element of y; x is an element of y according to all possibilities. Conversely, suppose our interpretation function assigns the empty set of possibilities to $\langle x, y\rangle$: that is, $I_\in(\langle x, y\rangle) = \emptyset$. Then according to that assignment, the claim that x is a member of y is *impossible.* No matter how things had been, x would not have been an element of y; x is an element of y according to no possibilities.

We can think of a set of possibilities as a *proposition*: the proposition that one of those possibilities obtains. Interpretation functions can therefore be seen as assignments of propositions to ordered pairs over the domain, which can then be extended in a natural and compositional way to assignments to each claim in the language. In particular, we let the possibilities for a disjunction be the possibilities for its disjuncts; the possibilities for an existentially quantified claim, the possibilities for its instances; and the possibilities for the negation of ϕ, the possibilities that are *incompossible* with the possibilities for ϕ (where p is *compossible* with q when there is some possibility that has them both as parts—that is, some r for which $r \leq p$ and $r \leq q$—otherwise, they are incompossible). Formally, if $[\![\phi]\!] \subseteq \mathbb{P}$ is the proposition assigned to ϕ, the idea is that we let $[\![\phi \vee \psi]\!] = [\![\phi]\!] \cup [\![\psi]\!]$, $[\![\exists x\phi]\!] = \bigcup_{x\in D}[\![\phi]\!]$, and $[\![\neg\phi]\!] = \{p : p \text{ is incompossible with every possibility in } [\![\phi]\!]\}$.

Propositions come equipped with a natural notion of implication. For propositions $X, Y \subseteq \mathbb{P}$, we say that X implies Y—in symbols, $X \vDash Y$—when it is impossible for X to be true but Y false. That is, when there is no possibility that has an X possibility *and* a $[\![\neg Y]\!]$ possibility as parts.[35] It is then straightforward to show that the laws of classical first-order logic without identity are preserved under implication: that is, if an argument from $\phi_0, ..., \phi_n$ to ψ is valid in first-order logic without identity, then $[\![\phi_0 \wedge ... \wedge \phi_n]\!] \vDash [\![\psi]\!]$.[36] Moreover, by picking a suitable interpretation function $I_=$, we can also ensure that the identity axioms are necessary so that all arguments of classical first-order logic are preserved under implication.[37] In what follows, I will assume that we have done so.

In forcing, we focus almost exclusively on a very specific kind of intensional model, one that is uniquely characterised by three core properties: Extensionality, Maximality, and Well-foundedness. Extensionality simply says that the axiom of extensionality is necessarily true in the model. Maximality says that the model contains as many possible sets as possible. More precisely, it says that given any set of possible sets X and any function f from those sets to propositions, there is a set s in D for which $[\![x \in s]\!]$ is implicationally equivalent to $[\![(x = z_0 \wedge f(z_0)) \vee (x = z_1 \wedge f(z_1)) \vee ... \vee (x = z_i \wedge f(z_i))...]\!]$ for $z_i \in X$. In other words, there is a set s such that to be an element of s just is to be equal to some z in X in so far as its

[35] Formally: $\neg\exists p\exists q, r(p < q, r \wedge q \in X \wedge r \in [\![\neg Y]\!])$. In other words, if any possibility containing an X possibility can be extended to contain a Y possibility. Formally: $\forall p \in X\forall q(q < p \rightarrow \exists r < q\exists s \in Y(r < s))$.

[36] Indeed, the propositions form a complete Boolean algebra relative to the implication relation (and modulo implicational equivalence). So, I will sometimes talk about infinite conjunctions and disjunctions.

[37] The simplest such function takes identity to be absolute, so that when $x = y$, $I_=(\langle x, y\rangle) = \mathbb{P}$ and when $x \neq y$, $I_=(\langle x, y\rangle) = \emptyset$. Unfortunately, this function will not always work for intensional models of set theory. In those models, we want the axiom of extensionality to be necessary. Given the necessity of the identity axioms, this means that $[\![\forall z(z \in x \leftrightarrow z \in y)]\!]$ will have to be implicationally equivalent to $[\![x = y]\!]$. It follows that variation in membership across possibilities will in general require variation in identities. Since the most interesting intensional models will have lots of variation in membership across possibilities, they will in general lots of variation in identities. It turns out that the hardest task in constructing the forcing model described below is precisely in obtaining this link between membership and identity.

associated proposition is true.[38] When membership in s is given by X in this way, I will say that X is a *core* for s.[39]

Well-foundedness is a bit trickier. It will help to consider an example. By Maximality, there is a possible set, $s_\emptyset$, that never contains anything: it is necessarily empty. Now suppose we have two incompossible possibilities p and q. Then, by Maximality, there will be two possible sets s_1 and s_2 such that s_1 contains $s_\emptyset$ in so far as p is the case (and otherwise contains nothing) and such that s_2 contains $s_\emptyset$ in so far as q is the case (and otherwise contains nothing). So, by Extensionality, according to p, $s_\emptyset$ is an element of s_1 and $s_2 = s_\emptyset$ and according to q, $s_\emptyset$ is an element of s_2 and $s_1 = s_\emptyset$. Now there are two importantly different ways of describing s_1 and s_2. On one description, s_1 has $\{s_2\}$ as a core and s_2 has $\{s_1\}$ as a core. To be an element of s_1 is to be equal to s_2 in so far as p is the case and to be an element of s_2 is to be equal to s_1 in so far as q is the case. Accordingly, they are non-well-founded: s_1 is a member of the core $\{s_1\}$ of s_2 and s_2 is a member of the core $\{s_2\}$ of s_1. On another description, they both have $\{s_\emptyset\}$ as a core. To be an element of s_1 is to be equal to $s_\emptyset$ in so far as p is the case and to be an element of s_2 is to be equal to $s_\emptyset$ in so far as q is the case. Accordingly, they are well-founded: each has $\{s_\emptyset\}$ as its core, and $s_\emptyset$ has nothing in its core, since its core is $\emptyset$. Well-foundedness says that the possible sets are well-founded under *some* such description, under some uniform allocation of cores to possible sets. More precisely, it says that there is a function F from D to cores such that there is no sequence $x_0, ..., x_n, ...$ of possible sets for which $F(x_n) \ni x_{n+1}$. We can then prove in ZFC that any two (definable) intensional models over the same space of possibilities that satisfy Extensionality, Maximality, and Well-foundedness are isomorphic (modulo necessary identity).[40]

The fundamental result in the method of forcing is that given any partial order $\mathbb{P}$, there *is* a (definable) intensional model over $\mathbb{P}$ satisfying Extensionality, Maximality, and Well-foundedness. Since this model is unique up to isomorphism (modulo necessary identity), I will refer to it as the *the forcing model* (for $\mathbb{P}$).[41] It is tedious but straightforward to verify

[38]Compare this with the requirement on classical models that for any subset X of the domain, there is a set in the model whose elements are precisely the members of X.

[39]It turns out that the notion of a core can be characterised without reference to f because we can show that X is a core for s (relative to any f) precisely when $[\![x \in s]\!]$ is implicationally equivalent to $[\![(x = z_0 \wedge z_0 \in s) \vee (x = z_1 \wedge z_1 \in s) \vee ... \vee (x = z_i \wedge z_i \in s)...]\!]$ for $z_i \in X$. A core for s is thus just a set, membership of whose elements in s determines membership of any possible set in s.

[40]In other words, if we modify the models so that necessarily identical sets are identical—that is, sets which are completely indistinguishable in the models—we get isomorphic models. *Proof sketch:* Let $\mathcal{M} = \langle D, I_=, I_\in, F\rangle$ and $\mathcal{M}' = \langle D', I'_=, I'_\in, F'\rangle$ be (definable) models over $\mathbb{P}$ that satisfy Extensionality, Maximality, and Well-foundedness, and let $\mathcal{M}_=$ and $\mathcal{M}'_=$ be the models we get by taking equivalence classes of necessarily identical sets in each. (Since these equivalence classes will be *proper* classes, we can use the Scott trick to obtain sets that play the same role.) Let $[\![\phi]\!]_{\mathcal{M}_=}$ be the proposition assigned to ϕ according to $\mathcal{M}_=$. Now, by a simple induction on the well-founded relation "$x \in F(y)$" we can use the constraints to recursively build a one-one function π from $\mathcal{M}_=$ to $\mathcal{M}'_=$ for which $[\![x = y]\!]_{\mathcal{M}_=}$ is implicationally equivalent to $[\![\pi(x) = \pi(y)]\!]_{\mathcal{M}'_=}$ and $[\![x \in y]\!]_{\mathcal{M}_=}$ is implicationally equivalent to $[\![\pi(x) \in \pi(y)]\!]_{\mathcal{M}'_=}$. This, in turn, extends via the compositional clauses for assigning propositions to show that $[\![\phi(\vec{x})]\!]_{\mathcal{M}_=}$ is implicationally equivalent to $[\![\phi(\vec{\pi(x)})]\!]_{\mathcal{M}'_=}$. $\square$. This theorem corresponds to the internal categoricity result in ZFC which says that any two (definable) well-founded models that satisfy extensionality and contain a set corresponding to each of their subsets are isomorphic. It is also straightforward to obtain analogues for intensional models of the usual quasi-categoricity theorems—which say that for any two models of ZFC2, one is isomorphic to an initial segment of the other—along the same lines.

[41]The usual way to construct the forcing model is relatively straightforward. Its domain is formed in a series of stages that are the intensional analogues of the V_αs. We start at stage 0 with no possible sets whatsoever. Then, at stage 1, we have all the functions from things at stage 0 to propositions. Since there is nothing

that in any intensional model satisfying Extensionality, Maximality, and Well-foundedness—and thus in any forcing model—the axioms of ZFC are necessary. And depending on what $\mathbb{P}$ we decide to work with, various other claims can be made necessary too. For example, there is a $\mathbb{P}$ relative to which CH is necessarily false.

The interesting possible sets are those for which membership is contingent. The witnesses to failures of CH, for example, must have at least some of their members contingently. But no matter how many interesting possible sets there are, there will also always be a wide range of boring possible sets for which membership is necessary. And it turns out that the possible which are transitively boring in this way—membership in them is a necessary matter, membership in their possible members is a necessary matter, membership in the possible members of their possible members is a necessary matter, etc—are isomorphic (modulo necessary identity) to the sets. The empty set, for example, corresponds to the possible set, $s_\emptyset$, that is necessarily empty; the singleton of the empty set corresponds to the possible set $s_{\{\emptyset\}}$, that necessarily contains $s_\emptyset$ and nothing else; and so on.[42] This means that in addition to *de dicto* necessity claims, we can also make sense of *de re* claims about sets, via the possible sets they correspond to. For example, let x be the powerset of ω and let s_x be the corresponding boring possible set. Then there is a space of possibilities $\mathbb{P}$ over which it is necessary in the forcing model that there is some subset of ω *not* in x, which is to say the proposition $[\![\exists y \subseteq \omega(y \notin s_x)]\!]$ is implicationally equivalent to $\mathbb{P}$. Similarly, given any set x, there is a $\mathbb{P}$ over which x is necessarily countable in the forcing model, which is to say the proposition $[\![s_x \text{ is countable}]\!]$ is implicationally equivalent to $\mathbb{P}$.

Since universes satisfy the axioms of ZFC, all of the above results can be carried out within them. In particular, the fundamental result of forcing implies that for any partial order $\mathbb{P}$ in any universe $\mathcal{U}$, the forcing model for $\mathbb{P}$ is definable within $\mathcal{U}$. Since forcing models can be seen as specifying a range of ways the universe of sets could be, a forcing model in $\mathcal{U}$ can be seen as specifying a range of ways $\mathcal{U}$ could be. What takes the width potentialist beyond this is the claim that some universe of sets $\mathcal{U}'$ *is* one of those ways. So let me now make this crucial claim precise.

Possibilities are like parts of possible worlds. It is therefore natural to think of a possible world as a set of possibilities: namely, the possibilities that are its parts. At a minimum, this set of possibilities should have two features. First, it should be rich enough to witness every proposition or its negation. Possible worlds are complete specifications of ways the world could be and the possibilities that compose it should witness this fact. Second, they

at stage 0, that means the only possible set at stage 1 is the empty set: the function that maps nothing to no propositions. At stage 1 we have all the functions from things at stage 1 to propositions. And so on. In general, stage $\alpha + 1$ comprises all and only the functions from things at stage α to propositions, stage λ comprises all and only the things at previous stages, and the possible sets of the model are all and only the things at some stage or other. By the way they're constructed, the possible sets of the model effectively have their cores "built in": the core of a function g at a stage is simply its domain. In particular, given a suitable $I_=$, we can explicitly define $I_\in(\langle f, g\rangle)$ to be $[\![(f = z_0 \wedge g(z_0)) \vee (f = z_1 \wedge g(z_1)) \vee ... \vee (f = z_i \wedge g(z_i)) \vee ...]\!]$ for $z_i \in \mathsf{dom}(g)$. This immediately guarantees both Maximality and Well-foundedness. So the only non-trivial task is to construct a suitable $I_=$ that will guarantee both Extensionality and the identity axioms. This is done by a subtle recursion. In particular, once we've defined $I_=$ for things in the domains of f and g, we define $[\![f = g]\!]$ to be the proposition that everything in the domain of f is a member of g in so far as it is a member of f and vice versa. We then carefully check that this indeed validates Extensionality and the identity axioms.

[42]In general, we can construct these boring possible sets recursively. In particular, the boring possible sets at stage $\alpha + 1$ are all and only the functions from boring possible sets at stage α to the propositions $\emptyset$ and $\mathbb{P}$. We can then recursively associate a unique boring possible set with each set in the obvious way.

should be modest enough not to witness some proposition and its negation. Possible worlds are ways the world could be, and the world could not be inconsistent. The possibilities that compose it should witness this fact too. Formally, for a universe $\mathcal{U}$ and partial order $\mathbb{P} \in \mathcal{U}$, say that a subset of $\mathbb{P}$ $X \in \mathcal{U}$ is a *possible world* (for $\mathbb{P}$ and $\mathcal{U}$) when it is both *complete*—for any proposition $Y \in \mathcal{U}$, X either contains a possibility in Y or a possibility in $[\![\neg Y]\!]$—and *consistent*—there is no proposition $Y \in \mathcal{U}$ such that X contains a possibility in Y and a possibility in $[\![\neg Y]\!]$.[43] Let W be a possible world for $\mathbb{P}$ and $\mathcal{U}$ and let $\mathcal{M}$ be the forcing model in $\mathcal{U}$ over $\mathbb{P}$. Say that a proposition $X \in \mathcal{U}$ is *true at* W just in case W contains a possibility p which implies X—that is, just in case $\exists p \in W(\{p\} \vDash X)$. We can then use W to define a classical model $\mathcal{M}_W$ from $\mathcal{M}$ in the obvious way. We let the domain of $\mathcal{M}_W$ be the possible sets of the forcing model—that is, we let the domain of $\mathcal{M}_W$ be the same as the domain of $\mathcal{M}$, i.e. D—and we stipulate that $\mathcal{M}_W \vDash x \in y$ iff $[\![x \in y]\!]$ is true at W and $\mathcal{M}_W \vDash x = y$ iff $[\![x = y]\!]$ is true at W. A simple induction on the complexity of ϕ establishes that ϕ is true in $\mathcal{M}_W$ iff it is true at W. Formally:

$$\mathcal{M}_W \vDash \phi \leftrightarrow [\![\phi]\!] \text{ is true at } W$$

The central width potentialist claim can then be formulated precisely as follows.

> Let $\mathcal{U}$ be a universe, $\mathbb{P}$ a partial order in $\mathcal{U}$, and $\mathcal{M}$ the forcing model over $\mathbb{P}$ in $\mathcal{U}$. Then there is a possible world W for $\mathbb{P}$ and $\mathcal{U}$ and a universe $\mathcal{U}'$ such that $\mathcal{U}'$ is isomorphic to $\mathcal{M}_W$ (modulo identity).

By working with the right partial orders, we can use this claim to obtain universes making all sorts of statements true. For example, the central width potentialist claim implies that as long as there is at least on universe, there is a universe making $\neg\mathsf{CH}$ true. Moreover, given the association of sets in $\mathcal{U}$ with the corresponding boring possible sets in the forcing model, we can show that for any $\vec{x} \in \mathcal{U}$, $[\![\phi(\vec{s_x})]\!]$ is true in W just in case $\phi(\vec{x})$ is true in $\mathcal{U}'$ and thus that $\mathcal{U} \subseteq \mathcal{U}'$.[44] It then follows that there are universes making all sorts of statements about the elements of $\mathcal{U}$ true. For example, let x be the powerset of ω in $\mathcal{U}$. Then the central width potentialist claim implies that there is a universe containing x in which some subset of ω is not in x. So, no universe contains absolutely all subsets of ω. Similarly, given any set x in $\mathcal{U}$, it implies that there is a universe $\mathcal{U}' \subseteq \mathcal{U}$ in which x is countable. So, every set in any universe is countable in some extended universe.[45]

As I mentioned in the introduction, the primary philosophical application of width potentialism is to the problem of independence. That problem, recall, is that some of the most fundamental questions in set theory are not answered by the axioms of ZFC and it is completely unclear how we might supplement those axioms to settle the questions in a well-motivated

[43] In the forcing literature, possible worlds in this sense are called *generic filters*. The definitions are different, but it is easy to see that they are equivalent.

[44] To prove this, we let i be the composition of the function from sets in $\mathcal{U}$ to the corresponding boring sets in the forcing model with the isomorphism from $\mathcal{M}_W$ to $\mathcal{U}'$. A simple induction then establishes that i is the identity function on $\mathcal{U}$.

[45] Given natural assumptions, this is actually equivalent to the central width potentialist claim. The reason is that when the powerset of $\mathbb{P}$ in $\mathcal{U}$ is countable, we can explicitly define a possible world for $\mathbb{P}$ and $\mathcal{U}$. And when that world exists in some $\mathcal{U}' \supseteq \mathcal{U}$, we can explicitly define a transitive subcollection $\mathcal{C} \subseteq \mathcal{U}'$ that is isomorphic to $\mathcal{M}_W$ (modulo identity). So if we assume that such collections also count as universes, then we have a witness to the central width potentialist claim.

way. The width potentialist's response has two parts: one negative and one positive. The negative part is that statements like CH are not unambiguous and well-defined. There is no ultimate background universe of sets, no ultimate V, in which we can query whether CH is true or not. The very questions that motivate the problem of independence, in other words, are illegitimate.

But what does it mean to say that there's no ultimate V? It will be helpful to look at some examples. Suppose it turned out that the collection of absolutely all sets—no matter what universe they come from, or even if they come from any universe at all—satisfied the axioms of ZFC. Clearly, that would count as an ultimate V. After all, it would comprise absolutely all sets and have all of the set theoretic structure required by the axioms of ZFC. In section 2, I pointed out that the core claim of set theory is the claim that every set is in some rank of the cumulative hierarchy: $\forall x \exists \alpha (x \in V_\alpha)$. So, if it turned out that the collection of absolutely all sets merely satisfied the axioms of, say, $\mathsf{Z} + \forall x \exists \alpha (x \in V_\alpha)$, I think we would still be inclined to think of them as an ultimate V. Or, suppose that there was a collection of sets satisfying the axioms of $\mathsf{Z} + \forall x \exists \alpha (x \in V_\alpha)$ that contained absolutely all of its subsets: that is, for every x in the collection and $y \subseteq x$, y is also in the collection. Then I think we would be inclined to say that it is an ultimate V *up to its height*. After all, the only sets it could possibly leave out would be those of rank higher than all of its ordinals. Since CH concerns sets in relatively small ranks of the cumulative hierarchy,[46] it would seem to get its ultimate and unambiguous formulation in each of these scenarios. Similarly, if it turned out that there was a set of absolutely all reals, a set of absolutely all sets of reals, and a set of absolutely all function from reals to reals, we would be inclined to think that CH gets its ultimate and unambiguous formulation in that context too. At the very least, then, in claiming that there is no ultimate V, the width potentialist should rule out these kinds of scenarios.

The positive part of the width potentialist's response to the problem of independence says that the versions of the fundamental questions—like whether CH is true—that *do* remain have answers. For example, the most salient version of the question whether CH is true is how it behaves across universes: is it true in all universes? in none? in some but not others? The central width potentialist claim provides a simple answer to these questions. As Hamkins puts it:

> On the [width potentialist] view, consequently, the continuum hypothesis is a settled question; it is incorrect to describe the CH as an open problem. The answer to CH consists of the expansive, detailed knowledge set theorists have gained about the extent to which it holds and fails in the multiverse, about how to achieve it or its negation in combination with other diverse set-theoretic properties.

If the width potentialist isn't careful, though, other equally legitimate versions of the fundamental questions may persist. For example, suppose that there are collections of sets that satisfy the axioms of ZFC or $\mathsf{Z} + \forall x \exists \alpha (x \in V_\alpha)$, that are in all other ways just like universes, but *aren't* counted as universes by the width potentialist. Then the question how CH behaves across universes appears to be as legitimate as the question how it behaves across these exouniverses, but the central width potentialist claim is silent on this latter question. The width potentialist should thus deny that there are such exouniverses. They should claim, in other words, that whatever looks for all intents and purposes like a universe of sets *is* a universe of sets. No universe left behind. Similarly, suppose that there are individual sets outside of any

[46]In particular, it can be coded as a statement in $V_{\omega+2}$.

universe—exosets. The question how CH behaves across universes appears to be as legitimate as the question how it behaves across exosets, but again the central width potentialist claim is silent on this. The width potentialist should thus also claim that there are no such sets, that absolutely every set is in some universe. No set left behind.[47]

In summary: I've argued that the width potentialist's central claim—that for any forcing model in $\mathcal{U}$ specifying a range of ways $\mathcal{U}$ could be, some universe is one of those ways—supports a response to the problem of independence only if it is supplemented with three other claims: (1) there is no ultimate V, no set of absolutely all reals, etc..., (2) whatever looks like a universe should be counted as a universe (no universe left behind), and (3) absolutely every set is in at least one universe (no set left behind).

4 Ultimate V

In this section I establish a simple but important result: the height potential sets are closed under their subsets. This result is then used to show that height potentialism is inconsistent with width potentialism. More positively, and independently of width potentialism, it is used to show that the height potential sets comprise an ultimate V *up to their height*. Its assumptions, moreover, are so general that they hold, suitably reinterpreted, for a broad class of other views. This means that the result can also be used to show that, for example, the iterative conception of sets is inconsistent with width potentialism and that the sets according to the iterative conception comprise an ultimate V up to their height. I will briefly discuss the significance of this at the end of the section.

I will work in the language of set theory supplemented with three modal operators: $\Diamond$, $\blacklozenge$, and @. We can think of $\Diamond$ and $\blacklozenge$ as expressing possibility in the height and width potentialist's sense respectively, and we can think of the actuality operator @ as *rigidly* referring to the actual circumstance. The actuality operator is added so as to provide a univocal point of evaluation for the other modal operators, and thus avoid unhelpful interactions between them.[48] I will therefore be interested in what's @$\Diamond$-possible and @$\blacklozenge$-possible rather than what's $\Diamond$-possible or $\blacklozenge$-possible.[49] I will use "could$_\Diamond$" and its cognates to express @$\Diamond$-possibility and

[47]Of course, by Gödel's incompleteness theorem, the width potentialist cannot completely avoid independence. Indeed, they face relatively interesting instances. For example, it's hard to see how they might settle the question whether there are universes where the axiom of choice fails but there are Reinhardt cardinals. The working hypothesis, presumably, is that these questions are either ultimately not as stubborn as whether CH holds, or not as important ([Hamkins et al., 2012, p.16]).

[48]Although natural, we need not think of the actuality operator as referring to the actual circumstance. The axioms I use for @ will hold whenever it expresses truth in some univocal point of evaluation for each of the other operators (see footnote ?). For example, for all those axioms say, it could express truth from the perspective of $V_1 = \{\emptyset\}$. The axioms, moreover, do not require that what's actual is possible in either sense. What's actual could be outside of what's possible. That is, they do not presuppose:

$$@\phi \rightarrow @\Diamond\phi$$

or:

$$@\phi \rightarrow @\blacklozenge\phi$$

[49]Of course, if @ expressed truth in the actual circumstance, then $@\Diamond\phi$ and $\Diamond\phi$ would be materially, though not necessarily, equivalent.

its cognates, and similarly for "could$_\blacklozenge$" and @$\blacklozenge$-possibility.

The background modal logic, $\mathsf{K}_@$, is a positive free version of K together with axioms ensuring the rigidity of @.[50,51] Let $\Diamond^*$ abbreviate the disjunction of $@\Diamond$ and $@\blacklozenge$. That is, let:

$$\Diamond^*\phi =_{df} @\Diamond\phi \vee @\blacklozenge\phi$$

We need three non-logical assumptions. Two concern the nature of sets. I will argue in the next section that each follows from the widely held thought that sets are completely characterised by what elements they have. The first is just the obvious generalisation of the axiom extensionality to the modal setting. It says that sets with the same possible$_{\Diamond^*}$ elements are identical. Formally:

(Extensionality$^\Diamond$) $$\Box^*\forall x\Box^*\forall y(\Box^*\forall z(\Diamond^*(z \in x) \leftrightarrow \Diamond^*(z \in y)) \rightarrow x = y)$$

The second says that sets cannot change their elements. Formally:

(Stability$^\Diamond$) $$x \in y \rightarrow \Box^*(Ey \rightarrow Ex \wedge x \in y)$$

The third and final assumption is an instance of the modal separation schema, which I argued in section 2 follows from height potentialism. Formally:

(Separation$^{\Diamond-}$) $$\Box^*\forall x@\Box\forall y@\Diamond\exists z[Ey \wedge \forall w(w \in z \leftrightarrow w \in y \wedge \Diamond^*(w \in x))]$$

Now I'm in a position to state and prove the promised result that the possible$_\Diamond$ sets are closed under their possible$_\blacklozenge$ subsets: that is, that every possible$_{\Diamond^*}$ subset of a possible$_\Diamond$ set is itself a possible$_\Diamond$ set.

[50] In particular, I will assume the following axioms for @.

$$@\neg\phi \leftrightarrow \neg@\phi$$

$$@\forall\mathbf{x}@\phi \leftrightarrow @\forall\mathbf{x}\phi$$

$$\Diamond@\phi \leftrightarrow @\phi$$

$$\blacklozenge@\phi \leftrightarrow @\phi$$

$$@\Box\forall\mathbf{x}@\Box\phi \rightarrow @\Box\forall\mathbf{x}\phi$$

$$@\blacksquare\forall\mathbf{x}@\blacksquare\phi \rightarrow @\blacksquare\forall\mathbf{x}\phi$$

where $\mathbf{x}$ is either a first-order or plural variable.

[51] A free quantificational logic is required because of the actuality operator. In classical logic, Ex is a theorem. By necessitation and universal generalisation, it follows that necessarily everything actually exists—$@\Box\forall x@Ex$—which is inconsistent with height potentialism.

Theorem 1 In $\mathsf{K}_@$, it follows from $\mathsf{Extensionality}^\Diamond$, $\mathsf{Stability}^\Diamond$, and $\mathsf{Separation}^{\Diamond-}$ that:[52]

$$(\mathsf{Subset\ Closure}^\Diamond) \qquad @\Box\forall x @\blacksquare\forall y(y \subseteq x \to @\Diamond Ey)$$

Although simple, theorem 1 has a number of important consequences. In the first instance, it shows that height and width potentialism are inconsistent. There are a number of different ways in which this manifests, but I will focus on what I take to be the most interesting three.

I argued in section 2 that the height potentialist should think that the height potential sets satisfy the axioms of $\mathsf{Z} + \forall x\exists\alpha(x \in V_\alpha)$. It follows from the arguments of section 3 that the width potentialist should count the height potential sets as a universe of sets. No universe left behind, not even a potential one. But the central width potentialist claim implies that for any (possible) universe of sets, there is a (possible) universe more subsets of the natural numbers. In the current terminology, this means that there could$_\blacklozenge$ be a subset of some possible$_\Diamond$ set that cannot$_\Diamond$ exist. Formally:

$$@\Diamond\exists x @\blacklozenge\exists y(y \subseteq x \land \neg @\Diamond Ey)$$

But that is just the negation of $\mathsf{Subset\ Closure}^\Diamond$.

Another inconsistency arises in virtue of the fact that the height potential sets satisfy the axiom of powerset. Let $\omega^\Diamond$ denote the possible$_\Diamond$ set of possible$_\Diamond$ natural numbers and let $\mathcal{P}^\Diamond(x)$ denote the possible$_\Diamond$ set of possible$_\Diamond$ subsets of x. It follows from $\mathsf{Subset\ Closure}^\Diamond$ that $\mathcal{P}^\Diamond(\omega^\Diamond)$ contains all possible$_\blacklozenge$ subsets of $\omega^\Diamond$. Formally:

$$@\blacksquare\forall x(x \subseteq \omega^\Diamond \to x \in \mathcal{P}^\Diamond(\omega^\Diamond))$$

As I argued in section 3, the width potentialist should think that absolutely any set is in at least one universe. No set left behind, not even a height potential one. So there should be some possible universe $\mathcal{U}$ containing both $\mathcal{P}^\Diamond(\omega^\Diamond)$ and $\omega^\Diamond$. The central width potentialist claim then implies that there is a (possible) universe in which there some subset of $\omega^\Diamond$ not in $\mathcal{P}^\Diamond(\omega^\Diamond)$. In the current terminology, that means:

$$@\blacklozenge\exists x(x \subseteq \omega^\Diamond \land x \notin \mathcal{P}^\Diamond(\omega^\Diamond))$$

Contradiction!

Finally, we can show that from the width potentialist's perspective, the height potential sets constitute an ultimate V, assuming that the height and width potentialist agree on

[52] At this point, I've made no assumptions about what possible$_\Diamond$ sets possibly$_\blacklozenge$ exist and vice versa. For all I've said, they might be completely disjoint. Because of this, $\mathsf{Subsets}$ may not be as strong as we'd like. To see this, suppose that when sets don't exist, they have no elements. Formally: $\neg Ex \to y \notin x$. Now if x doesn't possibly$_\blacklozenge$ exist, then it follows that its only possible$_\blacklozenge$ subset would be the empty set (assuming the empty set possibly$_\blacklozenge$ exists). But there could$_\blacklozenge$ still be non-empty subsets of x in the sense that their possible$_{\Diamond^*}$ elements are possible$_{\Diamond^*}$ elements of x. Formally, there could$_\blacklozenge$ be sets non-empty sets y for which:

$$\Box^*\forall z(z \in y \to \Diamond^*(z \in x))$$

Let $y \subseteq^* x$ abbreviate this claim. The corresponding version of $\mathsf{subsets}$ would then be:

$$@\Box\forall x @\blacksquare\forall y(y \subseteq^* x \to @\Diamond Ey)$$

It is easy to modify the proof of theorem 1 to show that this claim also follows from $\mathsf{Extensionality}^\Diamond$, $\mathsf{Stability}^\Diamond$, and $\mathsf{Separation}^{\Diamond-}$. Similarly, for any of the claims I make below, $\subseteq$ can be replaced with $\subseteq^*$.

the natural numbers.[53] Here's a sketch of the argument. The central width potentialist claim implies that any set in a possible universe is countable in some other possible universe. Formally:

$$@\blacksquare\forall x @\blacklozenge\exists f(f \text{ is a one-one function from } x \text{ to } \omega^{\lozenge})$$

It follows that any possible$_{\blacklozenge}$ set x can$_{\blacklozenge}$ be coded via its membership structure as a subset of the natural numbers. But theorem 1 tells us that any such code will possibly$_{\lozenge}$ exist. If the code is the code of a well-founded structure, then it is standard to use the axioms of $\mathsf{Z} + \forall x\exists\alpha(x \in V_\alpha)$ to obtain a set y with the same membership structure as x. It is then straightforward to use Extensionality$^{\lozenge}$ to show that $x = y$ after all. The upshot is that every possible$_{\blacklozenge}$ set which has a well-founded membership structure possibly$_{\lozenge}$ exists and thus that from the perspective of the width potentialist the height potential sets constitute an ultimate V. If we further assume that every set is in some universe—no set left behind—then the possible$_{\lozenge}$ sets will comprise absolutely all sets with a well-founded membership structure.[54]

Theorem 1 shows that height and width potentialism are jointly inconsistent. But it also has other important consequences. Although we took $\lozenge$ and $\blacklozenge$ to express height and width potentiality respectively, neither interpretation is mandatory, and by considering other interpretations, we can obtain a range of formally similar results. For example, theorem 1 shows that the height potential sets contain all of their possible$_{\blacklozenge}$ subsets on *any* interpretation of $\blacklozenge$ that obeys Extensionality$^{\lozenge}$ and Stability$^{\lozenge}$. So, if those principles are merely expressions of the nature of sets, then it seems like absolutely any set whatsoever should possibly$_{\blacklozenge}$ exist on at least *some* interpretation of $\blacklozenge$ that obeys them. It would then follow that the possible$_{\lozenge}$ sets

[53]Formally:

$$@\blacksquare\forall x(x \text{ is the first infinite ordinal} \rightarrow x = \omega^{\lozenge})$$

[54]Given these results, the width potentialist has two options. They either give up on their response to the problem of independence or they reject height potentialism. I want to note a residual worry for each option.

The standard forcing model according to which there are necessarily more subsets of the natural numbers is constructible in $\mathsf{Z} + \forall x\exists\alpha(x \in V_\alpha)$ (although, in this case, we won't be able to prove that it makes the axioms of ZFC necessarily but merely that it makes the axioms of $\mathsf{Z} + \forall x\exists\alpha(x \in V_\alpha)$ necessary. This means that we can define a forcing model within the height potential sets that specifies ways the height potential sets could be according to which there are more subsets of $\omega^{\lozenge}$, according to which there is some $x \subseteq \omega^{\lozenge}$ not in $\mathcal{P}^{\lozenge}(\omega^{\lozenge})$. Theorem 1 shows that no possible$_{\blacklozenge}$ can be like that. So the width potentialist is faced with an explanatory challenge, if they give up on their response to the problem of independence in order to salvage height potentialism: why do forcing models within possible universes specify other possible universes, but forcing models within the height potential sets do not?

Modal structuralism is a nominalist friendly form of height potentialism that translates claims in the language of set theory to claims about logically or metaphysically possible ZFC2 structures, where a structure is just some mereological sums of ordinary objects that behave suitably like set-theoretic ordered pairs (see Roberts [2019] for details). For example, on a metaphysical reading, collapse$^{\lozenge}$ becomes the claim that any things in any metaphysically possible ZFC2 structure metaphysically could have formed a set in some ZFC2 structure end-extending it. With suitable modifications, we can obtain a modal structural version of theorem 1. In particular, we can show that any possible$_{\blacklozenge}$ set whose members are all isomorphic to the members of some set in a possible ZFC2 structure is also isomorphic to a set in that same structure. In other words, we can show that the modal structural sets structurally comprise all of their possible$_{\blacklozenge}$ subsets. In a slogan: they comprise a structurally ultimate V (up to their height). Arguably, it follows that CH gets its ultimate and unambiguous formulation in its modal structural translation. The problem is that modal structuralism, understood metaphysically, is merely a thesis about how many ordinary objects are metaphysically possible; in particular, that there are enough to constitute the various ZFC2 structures. The width potentialist would thus seem to need to reject not just height potentialism, but modal structuralism, in order to salvage their response to the problem of independence. That makes it a hostage to metaphysical fortune.

comprise absolutely all of their subsets. Independently of width potentialism, then, theorem 1 shows that the height potential sets are an ultimate V *up to their height*: the only sets they miss out are those at possible$_\Diamond$ levels of the cumulative hierarchy indexed by impossible$_\Diamond$ ordinals. So, they comprise absolutely all subsets of the natural numbers, absolutely all countable ordinals, and absolutely all functions from the former to the latter. Given height potentialism, then, CH seems to get its ultimate and unambiguous formulation as the claim that it holds in the height potential sets.

We can also vary the interpretation of $\Diamond$. For example, suppose we interpret $\Diamond$ to mean @ and we let @ denote the actual circumstance as before. Suppose, moreover, that we adopt the iterative or limitation of size conceptions concerning the sets at the actual circumstance. Then each of Extensionality$^\Diamond$, Stability$^\Diamond$, and Separation$^{\Diamond-}$ will be as plausible as when $\Diamond$ was interpreted as expressing height potentiality. For example, Separation$^{\Diamond-}$ will come to the following instance of Separation.

$$\Box^*\forall x @\forall y\exists z\forall w(w \in z \leftrightarrow w \in y \wedge \Diamond^*(w \in x))$$

As I pointed out in section 2, Separation is an immediate consequence of both the iterative and limitation of size conceptions. Theorem 1 thus shows that both conceptions are just as inconsistent with width potentialism as height potentialism. Similarly, it shows that the iterative conception is committed to an ultimate V (up to the height of the stages). But the same light, height actualism will also be inconsistent with width potentialism.

This may strike you as strange. After all, height potentialism and the iterative and limitation of size conceptions represent our current best responses to the explanatory challenge raised by Russell's paradox. So width potentialism had better be consistent with one of them! The issue, as I see it, is that the inconsistency between height and width potentialism runs deeper than theorem 1 suggests. In particular, theorem 1 is downstream from a more fundamental disagreement about whether the explanatory challenge is a good one; about whether it requires an informative response. For the width potentialist, it must be a somewhat arbitrary matter which conditions determine sets (or determine sets within a given universe). For them, set theory is more like geometry (see, for example, Hamkins [2012]). Just as there is little more to a model of geometry over and above the fact that it satisfies some relevant axioms, so too there is little to a universe of sets over and above the fact that it satisfies some relevant fragment of ZFC.

5 Extensionality, stability, and separation

Can we resist theorem 1? To do so would mean rejecting either Extensionality$^\Diamond$, Stability$^\Diamond$, or Separation$^{\Diamond-}$. In this section, I will consider each in turn and argue that they should all be accepted by the height potentialist.

5.1 Extensionality and stability

Sets are completely characterised by what members they have. They are, in other words, extensional entities. One aspect of this idea is the thought that sets are *at least* characterised by what members they have. Once you know what elements a set has, you know what set it is; you can distinguish it from all other sets. Typically, this is formalised as the axiom of extensionality, the claim that sets with the same elements are identical.[55]

(Extensionality) $$\forall x, y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

But it also motivates Extensionality$^{\Diamond}$. After all, if possible$_{\Diamond^*}$ sets x and y have the same elements across worlds—that is, if $\Box^*\forall z(\Diamond^*(z \in x) \leftrightarrow \Diamond^*(z \in y))$—then the only way they could differ with respect to their members is not in *what* members they have, but *when* they have them. They would, in other words, be characterised intensionally rather than extensionally.

Another aspect is the thought that sets are *at most* characterised by what members they have, that there is *nothing more* to a set than what members it has.[56] Although Extensionality$^{\Diamond}$ tells us that the intensional membership behaviour of a set doesn't matter for its identity, it doesn't rule out that it has non-trivial intensional membership behaviour. For example, it is consistent with Extensionality$^{\Diamond}$ that the set containing the empty set, $\{\emptyset\}$, has $\emptyset$ as a member precisely when there is no greatest ordinal and otherwise contains nothing. Extensionality$^{\Diamond}$ merely requires that no other possible$_{\Diamond^*}$ set have only $\emptyset$ as a possible$_{\Diamond^*}$ element. The sensitivity of $\{\emptyset\}$ to whether there is a greatest ordinal would then be an essential feature beyond its having the elements it has. Stability$^{\Diamond}$ appears to be the most natural way to rule out this kind of non-trivial intensional membership behaviour and thus formalise the above thought.[57,58]

These two aspects are intimately connected. In fact, we can show that given plausible auxiliary assumptions, Stability$^{\Diamond}$ is derivable from Extensionality. Since the latter is universally accepted for sets, this provides yet more evidence in favour of Stability$^{\Diamond}$.[59] Here's the argument.

For generality, I will formulate Stability$^{\Diamond}$ with a single modal operator $\Box$ which I will assume is governed by a positive free version of K. The argument rests on two assumptions. The first is that the possible sets are closed under set subtraction: that whenever we have possible sets x and y, there could be a set which is exactly like x except that it cannot contain y. Let "$x\setminus\{y\}$" rigidly denote this set. The thought is that since $x\setminus\{y\}$ is completely specified in terms of x and y, it is a perfectly well-defined mathematical object; at least, as well-defined as x and y. By definition, $x \setminus \{y\}$ has the following two properties. It will be co-extensive with x except for y, whenever it and x both exist; and whatever it is, it does not contain y. Formally:

(i) $$\Box(E(x \setminus \{y\}) \wedge Ex \rightarrow \forall z(z \in x \setminus \{y\} \leftrightarrow z \in x \wedge z \neq y))$$

[55] Extensionality is universally adopted by philosophers working on modal set theory. See, for example, Linnebo [2013], Linnebo [2018], Studd [2019], Parsons [1983b], Fine [1981], and Reinhardt [1980].

[56] Linnebo, for example, says that "[o]nce you specify the elements of a set, you have specified everything that is essential to it". (p. 146, 2010). Of course, *being a set* is also essential to any set and does not in general depend on which elements it has. For example, the empty set has the same elements as any non-set. The claim is thus that once you have specified the elements of a set, you have specified everything that is essential to it *over and above its being a set.* See also [Fine, 1981, p. 179-180].

[57] It doesn't rule out all non-trivial intensional membership behaviour, however. In particular, it has nothing to say about the membership behaviour of a set in circumstances where it doesn't exist. See Roberts [forthcoming] for discussion of the same issue in the case of pluralities and for some natural ways to supplement Stability$^{\Diamond}$ to cover such cases.

[58] As with Extensionality$^{\Diamond}$, Stability$^{\Diamond}$ has also been unanimously adopted by philosophers working on modal set theory. See the references in footnote ?

[59] Boolos [1971] goes so far as to claim that Extensionality is close to an analytic truth. Of course, one person's modus ponens is another's modus tollens, and one might take my argument to show that in the modal setting, Extensionality is not as plausible as Extensionality$^{\Diamond}$ and in particular that it is not a consequence of the claim that sets are completely characterised by their members.

(ii) $$\Diamond(z = x \setminus \{y\}) \to y \notin z$$

The second assumption is that $x \setminus \{y\}$ is compossible with x in various ways. In particular, if x could exist without y or without containing y, then that fact should be compossible with the existence of $x \setminus \{y\}$. Formally:

(iii) $$\Diamond(Ex \wedge (\neg Ey \vee y \notin x)) \to \Diamond(Ex \wedge (\neg Ey \vee y \notin x) \wedge E(x \setminus \{y\}))$$

The thought here is that there is no relevant difference between x and $x \setminus \{y\}$. Since $x \setminus \{y\}$ cannot contain y, there seems to be no good reason why its existence would in any way depend on the existence of y nor of y's membership in x. And since it is exactly like x otherwise, there seems to be no good reason why it could not co-exist with x.

With these two assumptions in place, the argument proceeds as follows. For contradiction, suppose $\mathsf{Stability}^{\Diamond}$ fails. That is:

$$y \in x \wedge \Diamond(Ex \wedge (\neg Ey \vee y \notin x))$$

By (iii) and (i) we get:

$$y \in x \wedge \Diamond(Ex \wedge (\neg Ey \wedge y \notin x) \wedge E(x \setminus \{y\}) \wedge \forall z(z \in x \setminus \{y\} \leftrightarrow z \in x \wedge z \neq y))$$

Since the described possible scenario is one in which y either fails to exist or to be an element of x, it is a possible scenario in which x and $x \setminus \{y\}$ are co-extensive and thus by Extensionality identical. So:

$$y \in x \wedge \Diamond(x = x \setminus \{y\})$$

But by (ii), $y \in x$. Contradiction.[60]

Before I move on, let me briefly mention a slight modification of this argument for a claim I will appeal to in the next section. Above, I assumed that whatever $x \setminus \{y\}$ is, it doesn't contain y. But, of course, it is just as plausible, given our definition of $x \setminus \{y\}$ that whatever it is, it *can't* contain y. Formally:

(ii*) $$\Diamond(z = x \setminus \{y\}) \to \Box(y \notin z)$$

Then the same argument can be used to establish

($\mathsf{Stability}^{\Diamond *}$) $$\Diamond(y \in x) \to \Box(Ex \to (Ey \wedge y \in x))$$

by applying (ii*) instead of (ii).

[60]Instead of arguing for $\mathsf{Stability}^{\Diamond}$, as long as the space possibilities is rich enough, we can also make it true by definition along the following lines. Suppose that every possible$_{\Diamond}$ set exists in some possible$_{\Diamond}$ world where its elements all exist and are its elements. Similarly, for every possible$_{\blacklozenge}$ set. Now let $\Diamond'$ and $\blacklozenge'$ be $\Diamond$ and $\blacklozenge$ restricted to such worlds. It is easy to see that all of the principles other than set stability will hold good when formulated in terms of them. In particular, $\mathsf{collapse}^{\Diamond}$ is as plausible for $@\Diamond'$ as it was for $@\Diamond$. But now, set stability for $\Box^{', *}$ will be true by definition.

5.2 Separation

As I argued in section 2, to give up on $\mathsf{Separation}^{\Diamond}$ is to give up on the height potentialist's simple response to the challenge raised by Russell's paradox. If $\mathsf{Separation}^{\Diamond}$ fails, then set existence isn't merely a matter of co-existence. In plural terms: if $\mathsf{Separation}^{\Diamond}$ fails, then either plural existence isn't merely a matter of co-existence or some possible things might not have formed a set. Suitably adjusted, those arguments work equally well for $\mathsf{Separation}^{\Diamond*-}$. So the question is whether the height potentialist can weaken their response enough to avoid these arguments. As far as I'm aware, the only strategy that has been proposed is to retreat to the claim that set existence is a matter of co-existence *plus* a determinacy requirement. In plural terms: the suggestion is that plural existence is a matter of co-existence *plus* a determinacy requirement and set existence is a matter of the existence of a plurality. In the rest of this section, I will spell out the determinacy requirement and argue that it doesn't help to undermine the case for $\mathsf{Separation}^{\Diamond*-}$.

In section 2, I used the nothing over and above conception of pluralities—in particular, the idea of upward dependence–to motivate Plural Comp. The thought was that since each individual ϕ exists and since the ϕs taken together are nothing over and above the ϕs taken individually, there must be a plurality of all and only the ϕs. But this may have been too quick. Given objects $x_0, x_1, ..., x_i, ...$ etc, the nothing over and above conception tells us that there must be a plurality of them. But we might think it's a *further* claim that they are all and only the ϕs. Indeed, if they were all and only the ϕs, and y were a borderline ϕ, it would then seem to be indeterminate whether y is one of them. But it is not indeterminate whether y is x_0 or x_1 or... etc.[61] Generalising this thought, Yablo [2006], for example, claims:[62]

> The view once again is that plurality comprehension is mistaken.
>
> This may seem at first puzzling. The property P that (I say) fails to define a plurality can be a perfectly determinate one; for any object x, it is a determinate matter whether x has P or lacks it. How then can it fail to be a determinate matter what are all the things that have P? I see only one answer to this. Determinacy of the Ps follows from
>
> (i) determinacy of P in connection with particular candidates,
>
> (ii) determinacy of the pool of candidates.
>
> If the difficulty is not with (i), it must be with (ii). It is not the case that there are some things the [xxs] such that every candidate for being P is among them. If there were, one could go through the [xx]s one by one, asking of each whether it has P, thus arriving finally at the sought-after plurality of Ps.

Yablo's idea is that we need not accept an instance of Plural Comp if either the "pool of candidates" it comprehends on is somehow indeterminate or the condition on which it comprehends has indeterminate application.[63] Might this help us resist theorem 1? No.

[61] Assuming that identity and distinctness are determinate matters.

[62] Linnebo [2016] and Florio and Linnebo [2020] also deny instances of Plural Comp for similar reasons.

[63] It's important to note that although (i) and (ii) may be sufficient for the corresponding instance of Plural Comp, they aren't in general necessary. Simple supervaluational models show that Plural Comp for a predicate F is perfectly compatible with F having massively indeterminate application conditions and with there being

In $\mathsf{K}_{@}$, Separation$^{\lozenge*-}$ follows from Stability$^{\lozenge*}$, an analogous stability principle for pluralities that formalises the downward dependence idea, a suitably modified version of Collapse$^{\lozenge}$, and the following instance of Plural Comp (see theorem ? in the appendix).

(Plural Comp^{-}) $$\Box^*\forall x@\Box\forall y\exists xx\forall w(w \prec xx \leftrightarrow \lozenge^*(w \in y) \wedge \lozenge^*(w \in x))$$

To reject Separation$^{\lozenge*-}$, then, we have to reject Plural Comp^{-}. According to the proposal, we can do this only if either the "pool of candidates" for the relevant plurality is determinate or "$\lozenge^*(w \in y) \wedge \lozenge^*(w \in x)$" has indeterminate application. I will now argue that neither is the case.

Let Δ be a determinacy operator, which we can assume obeys the modal logic K. It follows that the compound operator $\Delta\Box^*$ also obeys K. The claim that "$\lozenge^*(w \in y) \wedge \lozenge^*(w \in x)$" has indeterminate application can be formalised as follows.

(*) $$\neg\Delta(\lozenge^*(w \in y) \wedge \lozenge^*(w \in x)) \wedge \neg\Delta\neg(\lozenge^*(w \in y) \wedge \lozenge^*(w \in x))$$

I claim that the argument I gave in section 5.1 for Stability$^{\lozenge*}$ generalises to the compound operator $\Delta\Box^*$. After all, Extensionality is something like an analytic truth; the first two assumptions of the argument are true in virtue of the definition of subtraction; and it's hard to see how we'd deny the compossibility assumption for this compound operator, given that we accepted it for $\Box^*$. So, I will assume:

(**) $$\neg\Delta\neg\lozenge^*(x \in y) \rightarrow \Delta\Box^*(Ey \rightarrow (Ex \wedge x \in y))$$

Given the right-hand-side of (*), (**) implies:

(***) $$\Delta\Box^*(Ey \rightarrow w \in y)$$

and:

(****) $$\Delta\Box^*(Ex \rightarrow w \in x)$$

---

a whopping amount of indeterminacy in what exists. Failures of (i) and (ii) *are* inconsistent with the following strengthened version of Plural Comp (assuming $\prec$ is determinate).

(Plural Comp$^{\Delta}$) $$\exists xx\Delta\forall x(x \prec xx \leftrightarrow \phi)$$

But, in classical logic, there are various F for which the corresponding instance of Plural Comp will be provable even though the corresponding instance of Plural Comp$^{\lozenge}$ is false. For example, let $x_0, ..., x_n$ be a sorites series for F. Classical logic ensures that there is a cut-off point in the series so that its Fs are either all and only $x_0, ..., x_n$ or all and only $x_1, ..., x_n$ or all and only $x_2, ..., x_n$, and so on. Assuming identity and distinctness are determinate, every condition of the form "$x = x_i \vee x = x_{i+1} \vee ... \vee x = x_n$" has determinate application conditions. Similarly, since its pool of candidates comprises the elements of the finite series $x_0, ..., x_n$, that too is plausibly determinate. By Yablo's lights, therefore, Plural Comp should hold for each of those conditions. Putting this all together, we get that there are some things which are all and only the Fs in the series.

$$\exists xx\forall x(x \prec xx \leftrightarrow x \text{ is an } F \text{ in the series})$$

But of course, there are no things that are determinately all and only the Fs. In classical logic, then, it won't always be the case that Plural Comp holds for ϕ precisely when (i) and (ii) hold for ϕ. So, (i) and (ii) don't give rise to a general account of plural existence, which is ultimately what we want.

In non-classical settings, however, it may be that (i) and (ii) are necessary for Plural Comp. For example, in intuitionistic logic it is natural to define determinacy in terms of excluded middle so that for $\prec$ to be determinate is for it to be the case that $x \prec xx \vee x \not\prec xx$. An instance of plural comprehension for F would then imply in intuitionistic logic that F has determinate application conditions.

So by the left-hand-side of (*), it follows that:

$$\neg\Delta\Diamond^{*}Ey \vee \neg\Delta\Diamond^{*}Ex$$

In other words, it follows from the assumption that "$\Diamond^{*}(w \in y) \wedge \Diamond^{*}(w \in x)$" has indeterminate application that either it is not determinate that x possibly$_{\Diamond^{*}}$ exists or it is not determinate that y possibly$_{\Diamond^{*}}$ exists. But the possible$_{\Diamond^{*}}$ existence of x and y is a *presupposition* of Plural Comp^{-}. It would hardly change the upshot of theorem 1 if we restricted it to determinately possible$_{\Diamond^{*}}$ sets.

What about the pool of candidates? Might that be relevantly indeterminate? In the presence of Stability$^{\Diamond^{*}}$ we have:

$$\Box^{*}\forall y, w(\Diamond^{*}(w \in y) \leftrightarrow w \in y)$$

It follows immediately that Plural Comp^{-} is equivalent to:

$$\Box^{*}\forall x @\Box\forall y\exists xx\forall w(w \prec xx \leftrightarrow w \in y \wedge \Diamond^{*}(w \in x))$$

We thus only need to comprehend on the elements of y, which are a determinate pool of candidates.

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